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ON GEOMETRIC AND TOPOLOGICAL IMPLICATIONS OF WEAK RICCI CURVATURE LOWER BOUNDS

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Abstract. In these notes, I briefly touch upon some analytic, geometric and topological properties of metric measure spaces with weak lower Ricci curvature bounds that my collaborators and I have observed in recent years.

Key words and phrases: Ricci curvature, RCD spaces, Aleksandrov spaces, Bakry-Émery manifolds, Low dimensional characterization, finitely generated fundamental group, A flow tangent to Ricci flow, Spectral rigidity.

1. Introduction

Studying less regular distance structures than Riemannian manifolds actually dates back to Riemann himself. He alluded to the definition what we now call Finslerian manifolds. This is still in the realm of starting from geometric objects that are defined in terms of infinitesimals (i.e. Riemannian or Finslerian metrics), the microscopic realm. Another perspective would be to start from a metric space (perhaps equipped with a measure) and study important geometric quantities – such as curvature – by merely looking at the properties of the distance function and measure, this, in contrast, is the macroscopic point of view.

The latter approach in the Riemannian geometry is nicely tied to the former by comparison theorems; based on which, one then is able to make sense of metric spaces with weak sectional curvature bounds (mainly attributed to Aleksandrov and Cartan-Aleksandrov-Toponogov). Note that (pointed) Gromov-Hausdorff limits of manifolds with uniform sectional curvature bounds give rise to the aforementioned spaces with the same weak curvature bounds; so such spaces describe what happens at the boundary of existence of Riemannian manifolds with curvature bounds. If one also employs the measure transportation, one can get to the notions of metric measure spaces with weak lower (and also upper) Ricci curvature bounds (the so-called Sturm-Lott-Villani bounds).

In these notes, we are concerned with metric measure spaces with weak lower Ricci curvature bounds that are also infinitesimally Hilbertian (the Riemannian feature) also known as RCD spaces and will mention some analytic, geometric and topological rigidity properties that they share with good old Riemannian manifolds.

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2. Preliminaries

In what follows, the primary objects of study are triples $(\mathrm{X}, \mathrm{d}, \mathrm{m})$, a Polish metric space (complete and separable) that is geodesic equipped with a locally finite Borel measure. Also consider the space of probability measures that are absolutely continuous with respect to m and with finite second moment, $\mathcal{P}_2(\mathrm{X})$.

Note that L^2-Wasserstein distance between probability measures turns the space of all probability measures with finite second moments into a complete geodesic space. Recall relative entropy which is the negative of the well known Boltzmann entropy.

Definition 2.1 ((dimension-less) Curvature-dimension conditions)**.** *We say* X *satisfies* $\mathsf{CD}(K, \infty)$ *conditions if for any two probability measures μ_0 and μ_1, the relative entropy is K-convex along a geodesic (consisting of probability measures) connecting the two.*

All complete Riemannian manifolds with $\mathrm{Ric} \geq K$ satisfy $\mathsf{CD}(K, \infty)$ as well as all Finsler manifolds whose dimension-free weighted Ricci curvature is bounded below by K. There is also the more involved notion of $\mathsf{CD}(K, N)$ spaces (using Rényi entropy instead) that restricts the Hausdorff dimension of X to be bounded above by N. Such convexity properties and their ties to geometry were brought up to light in the seminal works of Sturm [11, 12] and Lott-Villani [8].

Recall by the Cheeger-Colding theory, nontrivial Finsler manifolds cannot arise as Ricci limit spaces and a characteristic property of non-trivial Finsler manifolds is that their $\mathcal{W}^{1,2}$ Sobolev spaces fail to be Hilbert spaces; thus, in order to single out only "Riemannian objects" (what could arise as limits of manifolds with uniform lower Ricci bounds) in the big class of $\mathsf{CD}(K, N)$ spaces, one needs to further impose the infinitesimal Hilbertianity condition and this gives rise to RCD metric measure spaces ("R" for Riemannian) i.e. spaces satisfying "CD" whose $\mathcal{W}^{1,2}$ Sobolev spaces are Hilbert. Making this rigorous, requires a careful definition and analysis of Cheeger-Dirichlet energy functional in this singular setting.

Let us just list some important facts about RCD spaces:

- Riemannian manifolds, Aleksandrov spaces, Bakry-Émery manifolds (also called smooth measure spaces) and products, direct limits, fixed point free quotients by isometries of such spaces are all examples of RCD spaces;
- RCD spaces are essentially non-branching (recently a proof of non-branching property is suggested);
- Among other involved calculus developed for such spaces, one can perform first and second order calculus on RCD spaces i.e. inner products of gradients and Hessians can be defined with very similar calculus rules as in Riemannian setting.
- RCD spaces characteristically enjoy the Bochner inequality (expressed using the square field operator and its second iteration), a feature that opens the door for the use of the maximum principle.

For further details regarding curvature-dimension conditions, we refer the reader to [8, 11, 12] and for a review of main geometric analytic tools in RCD, see [2]. For a very brief overview of the needed technology, see the beginning sections of [5].

3. Main results

3.1. **Milnor's question.** A well-known conjecture due to Milnor is that complete manifold with nonnegative Ricci curvature posses finitely generated fundamental group. This has only been completely settled in dimension 3 so far. The following result [6] generalizes – to RCD spaces – an affirmative answer to Milnor's conjecture in a special case that is due to Sormani [10].

Theorem 3.1 (Kitabeppu-L. '15)**.** *Let* $(\mathrm{X}, \mathrm{d}, \mathrm{m})$ *be a connected, locally contractible, and non-branching geodesic metric-measure space with* $\mathrm{supp}(\mathrm{m}) = \mathrm{X}$*. Suppose* X *satisfies the* $\mathsf{CD}(0, N)$ *curvature-dimension conditions that is also infinitesimally Hilbertian. If* X *has small linear diameter growth* $\limsup \frac{\mathrm{diam}\partial(B(p,r))}{r} < 4S_N$*, (where the values of* S_N *are calculated in* [6]*) Then,* X *has finitely generated fundamental group.*

Proof. The proof follows the footsteps of the proof in [10] however one needs to use a weak version of Abresch-Gromoll excess theorem in RCD spaces. □

3.2. **Characterization of low dimensional RCD spaces.** One quest in this field is to axiomatize Ricci limit spaces. There are related open problems due to Sturm and Villani.

Recall a point in X is called k-regular if all blow up limits around it are metric-measure isometric to $\mathbb{R}^k$ (compare with interior points of manifolds versus corner or edge points); all such points comprise the regular stratum $\mathcal{R}_k$.

One quite interesting fact is that below synthetic dimension $N = 2$, the picture is Riemannian i.e. $\mathsf{RCD}(K, N < 2)$ spaces are indeed one dimensional manifolds, this is a corollary of the following more general theorem proven in [7].

Theorem 3.2 (Kitabeppu-L. '16)**.** *Let* X *be an* $\mathsf{RCD}(K, N)$ *space for* $K \in \mathbb{R}$ *and* $N \in (1, \infty)$*. Assume X is not one point and* $\mathrm{supp}(\mathrm{m}) = \mathrm{X}$*. The following are equivalent:*

1. $\mathcal{R}_1 \neq \emptyset$,
2. $\mathcal{R}_j = \emptyset$ *for any* $j \geq 2$,
3. $\mathrm{m}(\mathcal{R}_j) = 0$ *for any* $j \geq 2$,
4. X *is isometric to* $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{S}^1(r)$ *or* $[0, \mathrm{diam}(\mathrm{X})]$.

Moreover, the measure m *is equivalent to the* 1*-dimensional Hausdorff measure* $\mathcal{H}^1$ *i.e.* m *can be written in the form* $\mathrm{m} = e^{-f}\mathcal{H}^1$ *for a* (K, N)*-convex function* f*.*

Proof. By a careful point picking argument and supposing X is not 1D, it is shown that the existence of a 1-regular point along with entropy convexity causes a positive worth of measures to branch passing through p thus contradicting the essential non-branching property. □

Remark 3.3. *Utilizing the above result (among other things), Lytchak-Stadler* [9] *showed that two dimensional* RCD *spaces are indeed Aleksandrov surfaces.*

3.3. **A flow tangent to Ricci flow.** An intrinsic flow for RCD spaces has been introduced by Gigli-Mantegazza [3] using heat flow and optimal transport. As for any geometric flow, the formation of singularity is the main question for this flow. The following result shows under this flow, the singular set does not grow [1].

Theorem 3.4 (Bandara-L.-Munn '17)**.** *Let* M *be a smooth, compact manifold with* rough metric g *that induces a distance metric* d_g*. Moreover, suppose there exists*

$K \in \mathbb{R}$ and $N > 0$ such that $(M, g, \mu_g) \in \mathsf{RCD}(K, N)$. If $\mathcal{S} \neq M$ is a closed set and $g \in \mathcal{C}^k(M \setminus \mathcal{S})$, there exists a family of metrics $g_t \in \mathcal{C}^{k-1,1}$ on $M \setminus \mathcal{S}$ evolving according to the GM flow on $M \setminus \mathcal{S}$. For two points $x, y \in M$ that are g_t-admissible, the distance $\mathrm{d}_t(x, y)$ given by the Gigli-Mantegazza flow is induced by g_t.

Proof. The proof entails a careful analysis of the continuity equation which – in this setting – is a divergence form pde with measurable coefficients. □

3.4. **Spectral rigidity (first gap).** After a few decades of being investigated, in 2007, the spectral rigidity (of Zhong-Yang eigenvalue bounds) for Ricci non-negatively curved manifolds was established in Hang-Wang [4]. The following is a generalization to RCD spaces [5].

Theorem 3.5 (Ketterer-Kitabeppu-L. '23)**.** *Suppose* X *is a compact* $\mathsf{RCD}\,(0, N)$ *space with* $\mathrm{supp}\,(\mathrm{m}) = \mathrm{X}$. $\lambda_1 = \frac{\pi^2}{\mathrm{diam}^2}$ *if and only if* X *is either a weighted circle or a weighted line segment; In either cases, the space is equipped with a constant weight function i.e.* $\mathrm{m} = c\mathcal{H}^1$ *(in other words, X is a non-collapsed one dimensional* RCD *space).*

Proof. Assuming the bound is achieved, an involved analysis of eigenvalue problem in the singular setting would yield a harmonic potential the gradient flow of which gives a 1D isometric foliation of the space. Then the diameter restriction would imply the space must be 1D. □

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ON L-REDUCIBLE FINSLER METRICS

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ABSTRACT. In this paper, we study one of the oldest open problems in Finsler geometry which was introduced by Matsumoto-Shimada in 1977 about the existence of a concrete L-reducible Finsler metric that is not C-reducible. To spot such a Finsler metric, we study the class of spherically symmetric Finsler metrics. We prove two rigidity theorems for spherically symmetric Finsler metrics. First, we prove that every spherically symmetric Finsler metric is semi-C-reducible. Second, we show that every non-Riemannian spherically symmetric Finsler metric is a generalized L-reducible metric. Finally, we prove that every non-Riemannian L-reducible spherically symmetric Finsler metric on a manifold of dimension $n \geq 3$ must be a Randers metric.

Key words and phrases: L-reducible metric; C-reducible metric; spherically symmetric Finsler metric; Randers metric; Landsberg metric.

1. Introduction

The class of Randers metrics is a valuable and important class of non-Riemannian Finsler metrics which was introduced in 1941 by Physician Gunnar Randers to study general relativity in 4-dimensional Riemannian manifolds [5]. His discovered metric is in the form of $F = \alpha + \beta$, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is gravitation field and $\beta = b_i(x)y^i$ is the electromagnetic field. In his research, Randers regarded this metric not as a Finsler metric but as "affinely connected Riemannian metric", which is a rather strange notion in Riemannian Geometry.

An interesting and fascinating reality about Randers metrics are hidden in its Cartan torsion discovered by Makoto Matsumoto in 1974. For an n-dimensional Finsler manifold (M, F), the third derivatives of $1/2F_x^2$ at $y \in T_xM_0$ is symmetric trilinear forms $\mathbf{C}_y$ on T_xM which is called by the Cartan torsion of F. Éli Cartan introduced this torsion to characterize Riemannian metrics from the Finsler metrics. Taking a trace of Cartan torsion $\mathbf{C}$ gives the mean Cartan torsion $\mathbf{I} := trace(\mathbf{C})$. In 1972, Matsumoto introduced the Matsumoto torsion as follows

(1.1)
$$\mathbf{M}_y(u,v,w) = \mathbf{C}_y(u,v,w) - \frac{1}{n+1}\Big\{\mathbf{I}_y(u)\mathbf{h}_y(v,w) + \mathbf{I}_y(v)\mathbf{h}_y(u,w) + \mathbf{I}_y(w)\mathbf{h}_y(u,v)\Big\},$$

where $\mathbf{h}_y(u,v) := \mathbf{g}_y(u,v) - F^{-2}(y)\mathbf{g}_y(y,u)\mathbf{g}_y(y,v)$ is called the angular form in direction y and $\mathbf{g}_y$ is the fundamental tensor of F [2]. A Finsler metric F on a manifold M of dimension $n \geq 3$ is called C-reducible if $\mathbf{M} = 0$. Matsumoto showed that every Randers metric is C-reducible [2]. After six years, in 1978, Matsumoto-Hōjō proved that the converse is true too, namely, a positive-definite Finsler metric F is C-reducible if and only if it is a Randers metric [4]. They called this result as *conclusive Theorem*.

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In 1975, the well-known Japananese Physicist Y. Takano developed the theory of fields in Finsler spaces, where the fields have internal freedom. In particular, he studied the spinor fields details and found it necessary to introduce the gauge fields into the spinor field equations. Takano studied the field equations in a Finsler manifold and proposed certain interesting geometrical problems [8]. He requested mathematicians to find some proper forms of Landsberg curvature from the standpoint of Physics. In 1978, Matsumoto introduced the notion of L-reducible Finsler metrics as an answer to Takano, which was a generalization of C-reducible Finsler metrics [1]. A Finsler metric F on an n-dimensional manifold M is L-reducible if its Landsberg curvature is given by

$$\mathbf{L}_y(u,v,w) = \frac{1}{n+1}\Big\{\mathbf{J}_y(u)\mathbf{h}_y(v,w) + \mathbf{J}_y(v)\mathbf{h}_y(u,w) + \mathbf{J}_y(w)\mathbf{h}_y(u,v)\Big\}, \tag{1.2}$$

where $\mathbf{J} := trace(\mathbf{L})$ denotes the mean Landsberg curvature of F. Throughout this paper, we exclude the trivial cases of L-reducible metrics [10], including Riemannian metrics and locally Minkowskian metrics.

As we mentioned, Matsumoto defined (1.2) when he studied the hv-curvature P_{ijkl} of the Cartan connection. Then, he called such Finsler metrics by the notion of *P-reducible* since it comes from the P-curvature and we call them here "L-reducible metrics" for the relation with Landsberg curvature. If $\mathbf{L} = 0$, then F is called the Landsberg metric [9]. We have concrete examples of non-Landsberg L-reducible Finsler metrics. For example, it is evident that every C-reducible metric is L-reducible. However, the converse of this fact may not be accurate in general. For a Finsler metric of dimension $n \geq 3$, Matsumoto found some conditions under which the Finsler metric will be L-reducible. Since the study of Landsberg curvature has become an urgent necessity for the Finsler geometry as well as for theoretical physics, Matsumoto-Shimada studied some of Riemannian and non-Riemannian curvature properties of L-reducible metrics in [3]. They introduced the following open problem:

Is there any L-reducible Finsler metric that is not C-reducible?

There is an interesting generalization of C-reducible metrics. A Finsler metric F on an n-dimensional manifold M is called quasi-C-reducible if its Cartan torsion is written as follows

$$\mathbf{C}_y(u,v,w) = \frac{1}{n+1}\Big\{\mathbf{I}_y(u)\mathbf{H}_y(v,w) + \mathbf{I}_y(v)\mathbf{H}_y(u,w) + \mathbf{I}_y(w)\mathbf{H}_y(u,v)\Big\}, \tag{1.3}$$

where $\mathbf{H}_y := H_{ij}dx^i dx^j$ is a symmetric tensor so that $H_{ij}y^i = 0$. In [3], Matsumoto-Shimada proved the following.

Theorem A. ([3]) Every 3-dimensional quasi-C-reducible Finsler metric is L-reducible if and only if it is C-reducible.

In [7], Shibata tried to find concrete L-reducible metrics in the class of Weyl metrics, i.e., Finsler metrics of scalar flag curvature. Then he proved the following.

Theorem B. ([7]) Let (M, F) be a Finsler manifold of dimension $n \geq 3$. Suppose that F is of non-zero scalar flag curvature. Then F is L-reducible if and only if it is C-reducible.

An (α,β)-metric is a scalar function on TM defined by $F := \alpha\phi(s)$, $s = \beta/\alpha$, in which $\phi = \phi(s)$ is a C^∞ function on $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}y^iy^j}$ is a Riemannian metric, $\beta = b_i(x)y^i$ is a 1-form on M and $b := \|\beta_x\|_\alpha$ (see [11, 13]). In [6], Shibata characterized L-reducible (α,β)-metrics and showed the following surprising fact.

Theorem C. ([6]) Every non-Riemannian (α,β)-metric on a manifold M is L-reducible if and only if it is C-reducible.

Then L-reducible (α,β)-metrics must be Randers or Kropina metrics. Taking into account Shibata' results, one can conclude that the problem of existence of a concrete L-reducible metric is becoming more and more puzzling.

Inspired by the Numata-type metrics, in [12], Sadeghi and the author introduced a new class of Finsler metrics that contains the class of L-reducible metrics. This class of metrics is called generalized P-reducible metrics. Then, we obtained the following.

Theorem D. ([12]) Every generalized P-reducible (α,β)-metric with vanishing S-curvature is a Berwald metric or C-reducible metric.

By Theorem D, it follows that there is no concrete L-reducible (α,β)-metric with vanishing S-curvature. This is a conclusion of Theorem C.

To find concrete L-reducible Finsler metrics, one can consider the class of regular spherically symmetric Finsler metrics. A Finsler metric F on a domain $\mathcal{U} \subseteq \mathbb{R}^n$ is called spherically symmetric metric if it is invariant under any rotations in $\mathbb{R}^n$. In this case, there exists a positive function ϕ depending on two variables so that F can be written as

$$F = |y|\phi\left(|x|, \frac{\langle x, y\rangle}{|y|}\right),$$

where x is a point in the domain $\mathcal{U}$, y is a tangent vector at the point x, $|x| = \sqrt{\sum_{i=1}^n (x^i)^2}$, $|y| = \sqrt{\sum_{i=1}^n (y^i)^2}$ and $\langle x, y\rangle = \sum_{i=1}^n x^iy^i$. Let us put $u := |y|$, $v := \langle x, y\rangle$, $r := |x|$ and $s :=< x, y > /|y|$. Then a spherically symmetric Finsler metric can be written as $F = u\phi(r, s)$. The geodesic spray coefficients of spherically symmetric Finsler metric $F = u\phi(r, s)$ is given by $G^i = uPy^i + u^2Qx^i$, where P and Q defined by ϕ, r and s. In [14], the class of L-reducible spherically symmetric Finsler metrics is studied, and the following is obtained.

Theorem E.([14]) Let $F = u\phi(r, s)$ be a spherically symmetric Finsler metric on a domain $\mathcal{U} \subseteq \mathbb{R}^n$. Then F is a L-reducible metric if and only if it satisfies the following PDE

$$(\phi - s\phi_s)\, L_1 - 3\phi_{ss}L_2 = 0, \tag{1.4}$$

where

$$\begin{aligned} L_1 &:= 3\phi_sP_{ss} + \phi P_{sss} + \big(s\phi + (r^2 - s^2)\phi_s\big)Q_{sss}, \\ L_2 &:= \phi_s(P - sP_s) - s\phi P_{ss} + \big(s\phi + (r^2 - s^2)\phi_s\big)(Q_s - sQ_{ss}). \end{aligned}$$

However, due to the incredible complexity, we did not get any chance to solve (1.4). Even the Maple program could not find any solution for it.

Consider the special form of mean Landsberg curvature $\mathbf{J}$ of spherically symmetric Finsler metrics, and define the following

$$\Xi := \frac{1}{n+1}\Big(h_{ij}s^is^j-(n+1)(r^2-s^2)\Big)J_ks^k+\frac{1}{c_2c_5}\Big(c_2h_{ij}s^is^j+c_4(r^2-s^2)^2\Big)\Omega, \tag{1.5}$$

where h_{ij} are the components of angular metric, c_2 and c_4 defined by s and Ω defined by ϕ and s, $s^i := s_i := x_i - sr_i$ and $r^i := r_i := y^i/u$. Here, we consider L-reducible spherically symmetric Finsler metrics and prove the following.

Theorem 1.1. *Let $F = u\phi(r,s)$ be a non-Riemannian regular spherically symmetric Finsler metric on a manifold M of dimension $n \geq 3$. Suppose that $\Xi \neq 0$. Then F is L-reducible if and only if it is a Randers metric.*

Here, we give some remarks. We must explain that the method used to prove Theorem 1.1 is independent of solving PDE (1.4). Also, every two-dimensional Finsler metric is C-reducible which is not necessarily a Randers metric. Then, we exclude the case $n = 2$ in Theorem 1.1. We notify that Theorem 1.1 can be considered a complement to Theorem E or even a natural extension of it. Also, Theorem 1.1 gives a negative answer to Matsumoto-Shimada's open problem in the class of spherically symmetric Finsler metrics. It is remarkable that, the equation $\Xi = 0$ gives us three non-linear ordinary differential equations (see Proposition 1.3). We have not been able to find any solution for these ODE's until now. But, we certainly believe that the solutions of the three non-linear ODE (1.6)-(1.8) in Proposition 1.3 cannot be expressed in terms of elementary functions.

Landsberg metrics are special L-reducible metrics. Here, we get the following.

Corollary 1.2. *Every non-Riemannian spherically symmetric Finsler metric of Landsberg-type on a manifold M of dimension $n \geq 3$ with $\Xi \neq 0$ reduces to a Berwald metric.*

$\Xi = 0$ is a non-linear ODE which is divided to three non-linear ordinary differential equations. We are going to prove the following result.

Proposition 1.3. *Let $F = u\phi(r,s)$ be a spherically symmetric Finsler metric on a domain $\mathcal{U} \subseteq \mathbb{R}^n$ $(n \geq 3)$. Then $\Xi = 0$ if and only if $\phi = \phi(r,s)$ satisfies one of the following*

$$(r^2-s^2)\phi\phi_{ss} + \phi(\phi - s\phi_s) = \frac{n+1}{3}, \tag{1.6}$$

(1.7)
$$(\phi-s\phi_s)\left[\phi\phi_{sss}+(n+1)\phi_s\phi_{ss}-\frac{n+1}{r^2-s^2}\Big(s\phi\phi_{ss}-(\phi-s\phi_s)\phi_s\Big)\right]-(n-2)s\phi\phi_{ss}\phi_{ss}=0,$$

$$\Pi\Big[s\phi\Psi + \Pi\phi_s\phi_s - K(\phi\phi_s - s\phi_s\phi_s - s\phi\phi_{ss})\phi\Big] + s^2\phi^2\phi_{ss}\phi_{ss} = 0, \tag{1.8}$$

where $\Psi := (\phi_s\phi_{ss} + \phi\phi_{sss})$, $\Pi(s) := \phi - s\phi_s$ and

$$K := \frac{1}{(r^2-s^2)s_{|0}}s_{m|0}s^m. \tag{1.9}$$

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NATURALLY REDUCTIVE FINSLER METRICS ON HOMOGENEOUS SPACES

DARIUSH LATIFI

Abstract. In this paper, we study naturally reductive $(\alpha, \beta)-$metrics on homogeneous manifolds. We show that naturally reductive $(\alpha, \beta)-$metrics arise only when α is naturally reductive and some conditions on ϕ is satisfied. We give an explicit formula for the flag curvature of naturally reductive $(\alpha, \beta)-$metrics.

Key words and phrases: (α, β)-metrics; naturally reductive metrics; flag curvature.

1. Introduction

The study of invariant structures on homogeneous manifolds is an important problem of differential geometry. Among the Riemannian homogeneous metrics the naturally reductive ones are the simplest kind. They have nice simple geometric properties, but still form a large enough class to be of interest. The notion of naturally reductive Riemannian metrics was first introduced by Kobayashi and Nomizu [5]. The naturally reductive spaces have been investigated by a number of authors as a natural generalization of Riemannian symmetric spaces. The definition of naturally reductive homogeneous Finsler spaces is a natural generalization of the definition of naturally reductive Riemannian homogeneous space. In literature, there are two version of the definition of naturally reductive Finsler metrics on a manifold. The first version was introduced by Deng and Hou in [4]. The second definition, was given by the author in [6].

Let $\alpha = \sqrt{\tilde{a}_{ij}(x)y^i y^j}$ be a Riemannian metric and $\beta(x, y) = b_i(x)y^i$ be a $1-$form on an $n-$dimensional manifold M. Let

$$\|\beta(x)\|_\alpha := \sqrt{\tilde{a}^{ij}(x)b_i(x)b_j(x)}. \tag{1.1}$$

Now, let the function F is defined as follows

$$F := \alpha\phi(s) \quad , \quad s = \frac{\beta}{\alpha}, \tag{1.2}$$

where $\phi = \phi(s)$ is a positive C^∞ function on $(-b_0, b_0)$ satisfying

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0 \quad , \quad |s| \le b < b_0. \tag{1.3}$$

Then by Lemma 1.1.2 of [3], F is a Finsler metric if $\|\beta(x)\|_\alpha < b_0$ for any $x \in M$.
A Finsler metric in the form (1.2) is called an $(\alpha, \beta)-$metric.
The 1-form β corresponds to a vector field $\tilde{X}$ on M such that

$$\tilde{a}(y, \tilde{X}(x)) = \beta(x, y). \tag{1.4}$$

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Also we have $\|\beta(x)\|_\alpha = \|\tilde{X}(x)\|_\alpha$ (for more details see [7, 1]). Therefore we can write $(\alpha, \beta)-$metrics as follows:

$$F(x,y) = \alpha(x,y)\phi(\frac{\tilde{a}(\tilde{X}(x),y)}{\alpha(x,y)}), \tag{1.5}$$

2. Main results

Definition 2.1. *A Riemannian homogeneous space $(G/H, g)$ is said to be naturally reductive if there exists a reductive decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ of $\mathfrak{g}$ satisfying the condition*

$$\langle [X,Y]_{\mathfrak{m}}, Z\rangle + \langle Y, [X,Z]_{\mathfrak{m}}\rangle = 0, \tag{2.1}$$

for all $X, Y, Z \in \mathfrak{m}$.

where $\langle , \rangle$ denotes the inner product on $\mathfrak{m}$ induced by the metric g [5]. The first version of definition of naturally reductive homogeneous Finsler was introduced by the S. Deng and Z. Hou in [4](see Remark 2.2 below).

Remark 2.2. *In [4], a homogeneous manifold G/H with an invariant Finsler metric F is called naturally reductive if there exists an invariant Riemannian metric g on G/H such that $(G/H, g)$ is naturally reductive and the connection of g and F coincide.*

In this definition, they assume that such a metric should be Berwaldian.
The second definition was given by the author in [7](see Definition 2.3 below).

The scheme is to treat the geometry of coset manifolds G/H as a generalization of the geometry of Lie group G (Since G/H reduces to G when H={e}). From this viewpoint, the isomorphism $\mathfrak{m} \simeq T_o(G/H)$ generalizes the canonical isomorphism $\mathfrak{g} \simeq T_eG$, and a G-invariant Riemannian metric on G/H generalizes a left-invariant metric on G. The notion of bi-invariant Riemannian metric on G generalizes as the notion of naturally reductive homogeneous Riemannian space.
In fact, when $H = \{e\}$, hence $\mathfrak{m} = \mathfrak{g}$, the condition (2.1) is just the condition

$$\langle [X,Y], Z\rangle + \langle Y, [X,Z]\rangle = 0, \tag{2.2}$$

for a bi-invariant Riemannian metric on G [8].

Definition 2.3 ([7])**.** *A homogeneous manifold G/H with an invariant Finsler metric F is called naturally reductive if there exists an $Ad(H)-$invariant decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ such that*

$$g_y([x,u]_{\mathfrak{m}}, v) + g_y(u, [x,v]_{\mathfrak{m}}) + 2C_y([x,y]_{\mathfrak{m}}, u, v) = 0 \tag{2.3}$$

where $y \neq 0, x, u, v \in \mathfrak{m}$.

Evidently this definition is the natural generalization of (2.1).

Theorem 2.4. *Let $(M = \dfrac{G}{H}, F)$ be a naturally reductive homogeneous Finsler space, where F is an invariant $(\alpha,\beta)-$metric defined by the Riemannian metric $\tilde{a} = \tilde{a}_{ij}dx^i \otimes dx^j$ and the vector field X such that $\phi'(r) \neq 0$. Then*
(a) Each geodesic of $(G/H, F)$ is an orbit of a one-parameter subgroup of isometries $\{\exp(tZ)\}, Z \in \mathfrak{g}$.
(b) $(G/H, F)$ is of Berwald type and the Chern connection of $(G/H, F)$ is given by $(\nabla_{Y^*}Z^*)_o = (-\dfrac{1}{2}[Y,Z]_{\mathfrak{m}})^*_o$ for all $Y, Z \in \mathfrak{m}$.

Let $(G/H, F)$ be a homogeneous Finsler manifold, where F is an invariant $(\alpha, \beta)-$metric defined by the invariant Riemannian metric $\tilde{a}$ and invariant vector field X. If $(G/H, F)$ is naturally reductive then $(G/H, \tilde{a})$ is naturally reductieve.

Theorem 2.5. *Let $(G/H, F)$ be a homogeneous Finsler space, where F is an invariant $(\alpha, \beta)-$metric defined by the Riemannian metric $\tilde{a} = \tilde{a}_{ij} dx^i \otimes dx^j$ and the vector field X which is parallel with respect to to $\tilde{a}$ and $\phi^{'}(r) \neq 0$, where $r = \dfrac{\tilde{a}(X, y)}{\sqrt{\tilde{a}(y, y)}}$. If $(G/H, \tilde{a})$ is naturally reductive, then $(G/H, F)$ is naturally reductive.*

Theorem 2.6. *Let $(G/H\ ,\ F)$ be a naturally reductive homogeneous Finsler space, where F is an invariant $(\alpha, \beta\ )-$ metric defined by the Riemannian metric $\tilde{a} = \tilde{a}_{ij} dx^i \otimes dx^j$ and the vector field X. Let (P, y) be a flag in $\mathfrak{m}$ such that $\{y, u\}$ is an orthonormal basis of P with respect to $\tilde{a}$. Then the flag curvature of (P, y) is given by*

$$K(P, y) = \frac{\phi(r) - \phi^{'}(r) r}{\phi^2(r) \psi} \left(\frac{1}{4} \parallel [u, y]_{\mathfrak{m}} \parallel^2 + \tilde{a}([[u, y]_{\mathfrak{h}}, u]_{\mathfrak{m}}, y) \right),$$

where $\parallel [u, y]_{\mathfrak{m}} \parallel$ denotes the norm of $[u, y]_{\mathfrak{m}}$ with respect to $\tilde{a} = \tilde{a}_{ij} dx^i \otimes dx^j$ and $r = \tilde{a}(X, y)$ and $\psi = \phi(r) + \phi^{''}(r) r^2 - \phi^{'}(r) r$.

Proof. Using the explicit expression for the connection of M, a straightforward lengthy calculation leads to the following expression for the curvature tensor R of $(G/H, F)$.

$$R_o(v, w)z = -[[v, w]_{\mathfrak{h}}, z] - \frac{1}{2}[[v, w]_{\mathfrak{m}}, z]_{\mathfrak{m}} - \frac{1}{4}[[z, v]_{\mathfrak{m}}, w]_{\mathfrak{m}} + \frac{1}{4}[[z, w]_{\mathfrak{m}}, v]_{\mathfrak{m}}, \tag{2.4}$$

for all $v, w, z \in \mathfrak{m} \cong T_o M$.
For the flag curvature we have

$$K(P, y) = \frac{g_y(R(u, y)y, u)}{g_y(y, y) g_y(u, u) - g_y^2(y, u)}. \tag{2.5}$$

The result is obtained by calculating and placing components in the curvature formula. □

Definition 2.7 ([2])**.** *A Finsler space with Finsler function*

$$F(x, y) = \alpha(x, y) + \beta(x, y)$$

is called a Randers space .

In [7], the author gives an explicit formula for the flag curvature of naturally reductive Randers spaces. As a corollary of Theorem 2.6, we have the following corollary.

Corollary 2.8. *Let $(G/H, F)$ be a naturally reductive Randers space with F defined by the Riemannian metric $\tilde{a} = \tilde{a}_{ij} dx^i \otimes dx^j$ and the vector field X. Let (P, y) be a flag in $\mathfrak{m}$ such that $\{y, u\}$ is an orthonormal basis of P with respect to $\tilde{a}$. Then the flag curvature of the flag (P, y) in $\mathfrak{m}$ is given by*

$$K(P, y) = \frac{1}{(1 + \tilde{a}(X, y))^2} \left(\frac{1}{4} \parallel [u, y]_{\mathfrak{m}} \parallel^2 + \tilde{a}([[u, y]_{\mathfrak{h}}, u]_{\mathfrak{m}}, y) \right).$$

We note that if the Randers space $(G/H, F)$ is Riemannian i.e. $X = 0$, then the above formula for flag curvature is just the formula for sectional curvature of a naturally reductive homogeneous Riemannian manifold [5, 7]:

$$K(u, y) = \frac{1}{4}\langle [u, y]_{\mathfrak{m}}, [u, y]_{\mathfrak{m}}\rangle + \langle [[u, y]_{\mathfrak{h}}, u]_{\mathfrak{m}}, y\rangle.$$

The following result generalizes Milnor results about the sectional curvature of bi-invariant Riemannian metrics (see [8]) to bi-invariant $(\alpha, \beta)-$metrics.

Theorem 2.9. *Let G be a Lie group with a bi-invariant $(\alpha, \beta)-$metric F defined by the Riemannian metric $\tilde{a} = \tilde{a}_{ij}dx^i \otimes dx^j$ and the vector field X such that $\phi^{'}(r) \neq 0$. Let (P, y) be a flag in $\mathfrak{g}$ such that $\{y, u\}$ is an orthonormal basis of P with respect to $\tilde{a} = \tilde{a}_{ij}dx^i \otimes dx^j$. Then the flag curvature of the flag (P, y) in $\mathfrak{g}$ is given by*

$$K(P, y) = (\frac{\phi(r) - \phi^{'}(r)\tilde{a}(X, y)}{4\phi^2(r)\psi}) \parallel [u, y] \parallel^2,$$

where $\parallel [u, y] \parallel$ denotes the norm of $[u, y]$ with respect to $\tilde{a} = \tilde{a}_{ij}dx^i \otimes dx^j$, and $r = \tilde{a}(X, y)$ and $\psi = \phi(r) + \phi^{''}(r)r^2 - \phi^{'}(r)r$.

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CRITICAL METRICS ON PSEUDO-RIEMANNIAN MANIFOLDS

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Abstract. Critical metrics are thoroughly studied in relation to quadratic curvature functionals over Gödel-type space-times. This study leads to the unequivocal determination of homogeneous critical metrics on the spaces being analyzed.

Key words and phrases: critical metric; Gödel space-time; quadratic curvature functional.

1. Introduction

In the field of differential geometry, a fascinating topic with significant applications in mathematical physics is the study of critical metrics over a family of (pseudo-)Riemannian manifolds. For an oriented, closed manifold M^n equipped with a family $\mathcal{M}_1$ of (pseudo-)Riemannian metrics of volume one, it is vital to determine extremum metrics from $\mathcal{M}_1$ for a specific curvature functional. This problem of critical metrics is of utmost importance.

Einstein metrics are widely recognized as critical metrics. The condition for Einstein metrics, $\varrho = \lambda g$ for some real constant λ, is the same as the Euler-Lagrange system related to the Einstein-Hilbert functional $g \mapsto \int_M \tau \mathrm{d}vol_g$. The Ricci tensor and scalar curvature, denoted by ϱ and τ respectively, are involved in this system.

Curvature functionals based on scalar quadratic curvature invariants have been extensively studied in the literature. This topic was initiated in the Riemannian settings in [1] and has been further explored by numerous scholars. For an in-depth survey on quadratic curvature functionals and critical metrics, we recommend referring to [2, 3], and the references therein. While quadratic curvature functionals can be studied in different dimensions of the base manifold, the dimension 4, which is the framework of space-times, receives more attention.

To study the quadratic curvature invariants, note that $\{\Delta\tau, \tau^2, ||\varrho||^2, ||R||^2\}$ serve as a basis. Thus, a generic functional on quadratic curvature invariants can be expressed as follows

$$g \mapsto \int_M \left(a\tau^2 + b||\varrho||^2 + c||R||^2\right) \mathrm{d}vol_g, \tag{1.1}$$

where a, b, c are arbitrary real constants and R is the curvature tensor. In dimension 4, the Gauss-Bonnet Theorem yields the equation

$$32\pi^2\chi(M) = \int_M \left(\tau^2 - 4||\varrho||^2 + ||R||^2\right) \mathrm{d}vol_g.$$

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This means that the critical points related to the curvature tensor are equivalent to the critical points of the functional $4||\varrho||^2 + \tau^2$. Therefore, the functional of equation (1.1) is equivalent to $g \mapsto \int_M \left((a-c)\tau^2 + (4c+b)||\varrho||^2\right) \mathrm{d}vol_g$. It is important to note that the manifold (M, g) is critical for the functional (1.1) whenever it is critical for the following functionals $\mathcal{S}$ and $\mathcal{F}_t$: $\mathcal{F}_t : g \mapsto \int_M \left(t\tau^2 + ||\varrho||^2\right) \mathrm{d}vol_g, \quad t \in \mathbb{R}$, and $\mathcal{S} : g \mapsto \int_M \tau^2 \mathrm{d}vol_g$.

It is absolutely imperative to note that for all quadratic curvature functionals $\mathcal{F}_t,\ t \in \mathbb{R}$, Einstein metrics are critical (see [2]). In the Riemannian settings, it has been established that for the functional $\mathcal{F}_{-1/3}$, the Bach-flat metrics are critical, while for both functionals $\mathcal{F}_{-1/4}$ and $\mathcal{S}$, the Weyl metrics with zero scalar curvature are critical. These well-known results certainly spark interest in exploring the critical metrics in different signatures for quadratic curvature functionals.

K. Gödel in [4] introduced homogeneous solutions to the Einstein's field equations with cosmological constant Λ for a universe in rotation ω and with an incoherent matter distribution, included the existence of closed time-like curves. These Gödel-type space-times, both in their homogeneous form and in higher dimensions, have been the subject of extensive study in Differential Geometry and Theoretical Physics.

The current study considers homogeneous Gödel-type space-times and explicitly determines classes of critical metrics for quadratic curvature functionals $\mathcal{S}$ and $\mathcal{F}_t$.

The upcoming section will present crucial facts and material necessary for examining critical metrics on Gödel-type space-times. Following that, the subsequent section will deliver the classification of critical metrics on the homogeneous Gödel-type space-times.

2. Preliminaries

Let $\mathcal{M}_1^n$ denote the set of (pseudo-)Riemannian metrics of volume one on a closed oriented manifold M^n. A real valued function F on $\mathcal{M}_1^n$, such that $F(\varphi^* g) = F(g)$ for every diffeomorphism φ and every $g \in \mathcal{M}_1^n$ is called a *(pseudo-)Riemannian functional*. Since φ is by definition an isometry between (M, g) and $(M, \varphi^* g)$, this means that the functional F only depends on Riemannian geometric data, and can be viewed as a function on the quotient space $\mathcal{M}_1/\mathfrak{D}$, where $\mathfrak{D}$ is the diffeomorphism group of M.

The Euler-Lagrange equations for a quadratic curvature functional are well established and can be computed in the Riemannian settings [1, 5]. It is worth noting that the results obtained for the Riemannian case can be effortlessly extended to the pseudo-Riemannian settings, as the arguments do not depend on the signature of the base metric.

For the functionals $\mathcal{F}_t : g \mapsto \int_M \left(t\tau^2 + ||\varrho||^2\right) \mathrm{d}vol_g$ and $\mathcal{S} : g \mapsto \int_M \tau^2 \mathrm{d}vol_g$, one can give the gradients as follows

$$
\begin{aligned}
(\nabla \mathcal{S})_{ij} &= 2\nabla^2_{ij}\tau - 2(\Delta\tau)g_{ij} - 2\tau\varrho_{ij} + \tfrac{1}{2}\tau^2 g_{ij}, \\
(\nabla \mathcal{F}_t)_{ij} &= -\Delta\varrho_{ij} + (1+2t)\nabla^2_{ij}\tau - \tfrac{1+4t}{2}(\Delta\tau)g_{ij} - 2t\tau\varrho_{ij} - 2\varrho_{kl}R_{ikjl} \\
&\quad + \tfrac{1}{2}\left(t\tau^2 + ||\varrho||^2\right) g_{ij}.
\end{aligned}
$$

Noting that, if $(\nabla \mathcal{F}_t) = cg$ for some real constant c, then g is critical for $\mathcal{F}_t$ and vice versa. By taking trace of the above equation we have

$$(n-4)\left(t\tau^2 + ||\varrho||^2\right) - (n + 4(n-1)t)\Delta\tau = 2nc.$$

Thus, g is critical for $\mathcal{F}_t$ if and only if

$$-\Delta\varrho_{ij}+(1+2t)\nabla^2_{ij}\tau-\frac{2t}{n}(\Delta\tau)g_{ij}-2\varrho_{kl}R_{ikjl}-2t\tau\varrho_{ij}+\frac{2}{n}\left(t\tau^2+||\varrho||^2\right)g_{ij}=0, \tag{2.1}$$

and

$$(n-4)\left(t\tau^2+||\varrho||^2-\lambda\right)=(n+4(n-1)t)\Delta\tau, \tag{2.2}$$

where $\lambda=\mathcal{F}_t(g)$ (see [3]). It is a logical outcome that $\mathcal{F}_t$ has critical points for Einstein metrics with any given value of t. Furthermore, as per the relation of $\nabla\mathcal{S}$ mentioned above, critical points for $\mathcal{S}$ are metrics with either vanishing scalar curvature or Einstein.

In spaces that are four-dimensional with constant scalar curvature, particularly in homogeneous spaces, the aforementioned Euler-Lagrange equations can be simplified significantly. In fact, in this scenario, equation (2.2) is satisfied automatically as $(\nabla\mathcal{S})_{ij}=2\tau\left(\frac{1}{4}\tau g_{ij}-\varrho_{ij}\right)$. Equation (2.1) is reduced to

$$\Delta\varrho+2R\left[\varrho\right]+2t\tau\varrho-\frac{1}{2}\left(||\varrho||^2+t\tau^2\right)g=0, \tag{2.3}$$

where $R\left[\varrho\right]$ is the tensor field defined by components $\varrho_{kl}R_{ikjl}$. The forthcoming sections will focus on this equation (2.3) and its solutions.

Scholars have extensively studied Gödel-type space-times. These space-times are exposed by the Lorentzian metrics using the local coordinate (t,r,ϕ,z).

$$g=[dt+H(r)d\phi]^2-dr^2-D^2(r)d\phi^2-dz^2, \tag{2.4}$$

The metrics are given by Equation (2.4), where (r,ϕ,z) are the usual cylindrical coordinates and $t\geq 0$ is the time variable and $r,\phi,z\in\mathbb{R}$ (undetermined for $r=0$). It is important to note that g is non-degenerate whenever $D(r)\neq 0$ since $\det(g)=-D^2(r)$. Furthermore, homogeneous Gödel-type space-times are manifolds that satisfy Equations (2.5), where ω and α are real scalars [7, 6].

$$D''=\alpha D,\quad H'=-2\omega D. \tag{2.5}$$

These manifolds have determining functions D and H that are of class $\mathcal{C}^\infty$, as opposed to at least $\mathcal{C}^2$ for the curvature tensor calculation, which is required for the functions D and H in Equation (2.4).

3. Homogeneous critical metrics

This section focuses on calculating critical metrics for the functionals $\mathcal{S}$ and $\mathcal{F}_t$ derived from homogeneous Gödel-type space-times. As stated in Section two, metrics with vanishing scalar curvature or Einstein are critical for the functional $\mathcal{S}$. It is important to note that the scalar curvature in the presence of the homogeneity condition (2.5) is easily $\tau=2(\alpha-\omega^2)$, meaning that the scalar curvature vanishes when $\alpha=\omega^2$. By applying this condition in equation (2.5), we deduce the following system of ordinary differential equations:

$$\begin{cases} H'+2\omega D=0,\\ D''-D\omega^2=0.\end{cases}$$

Through direct calculations, we obtain

$$H=c_1-2c_2e^{\omega r}+2c_3e^{-\omega r}\quad\text{and}\quad D=c_2e^{\omega r}+c_3e^{-\omega r},$$

where c_1,c_2,c_3 are arbitrary real constants. The following remark is a direct consequence of this arguments.

Remark 3.1. *A homogeneous Gödel-type space-time (M, g) is critical for the functional $\mathcal{S}$ if and only if either*

- $H = c_1, \quad D = c_2 r + c_3,$ *or*
- $H = c_1 - 2c_2 e^{\omega r} + 2c_3 e^{-\omega r}, \quad D = c_2 e^{\omega r} + c_3 e^{-\omega r},$

where c_1, c_2, c_3 are arbitrary real constants.

We must calculate critical homogeneous Gödel-types metrics for the functional $\mathcal{F}_t$. It is necessary to determine the tensors that create this relation according to equation (2.3). By considering the homogeneity conditions (2.5), expressions for the Levi-Civita connection, curvature tensor and the Ricci tensor are calculated directly as

$$\begin{aligned} &\nabla_{\partial_t}\partial_r = -\frac{\omega H}{D}\partial_t + \frac{\omega}{D}\partial_\phi, \quad \nabla_{\partial_t}\partial_\phi = -\omega D \partial_r, \\ &\nabla_{\partial_r}\partial_\phi = -\frac{\big(\omega(D^2+H^2)+HD'\big)}{D}\partial_t + \frac{(\omega H + D')}{D}\partial_\phi, \\ &\nabla_{\partial_\phi}\partial_\phi = -D\left(2\omega H + D'\right)\partial_r, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} R(\partial_t,\partial_r) =& -\omega^2\left(\partial_t \otimes dr + \partial_r \otimes dt + H\partial_r \otimes d\phi\right), \\ R(\partial_t,\partial_\phi) =& \omega^2\left(H\partial_t \otimes dt + (H^2-D^2)\partial_t \otimes d\phi - \partial_\phi \otimes dt - H\partial_\phi \otimes d\phi\right), \\ R(\partial_r,\partial_\phi) =& H(4\omega^2-\alpha)\partial_1 \otimes dr + H\omega^2\partial_r \otimes dt \\ &+ ((3\omega^2-\alpha)D^2 + H^2\omega^2)\partial_r \otimes d\phi + (\alpha - 3\omega^2)\partial_\phi \otimes dr, \end{aligned} \tag{3.2}$$

and

$$\varrho = 2\omega^2 dt^2 + 4\omega^2 H dt d\phi + (2\omega^2-\alpha)dr^2 + ((2\omega^2-\alpha)D^2 + 2\omega^2H^2)d\phi^2. \tag{3.3}$$

Through direct calculations and the application of the homogeneous condition (2.5) once more, it is evident that the Ricci tensor's Lapidarian can be expressed as follows:

$$\Delta\varrho = (\alpha - 4\omega^2)\left(4\omega^2 dt^2 + 8\omega^2 H dt d\phi + 2\omega^2 dr^2 + 2(2H^2+P^2)d\phi^2\right).$$

By utilizing the equations (3.2) and (3.3) and conducting a metric contraction on the indices, $\varrho_{kl}R_{ikjl}$, the tensor field $R[\varrho]$ can be calculated.

$$\begin{aligned} R[\varrho] =& 2\omega^2(\alpha-2\omega^2)dt^2 + 4H\omega^2(\alpha-2\omega^2)dt d\phi - (8\omega^4 - 5\alpha\omega^2 + \alpha^2)dr^2 \\ &- (D^2(8\omega^4 - 5\alpha\omega^2+\alpha^2) - 2H^2\omega^2(\alpha-2\omega^2))d\phi^2. \end{aligned}$$

The criticality of the homogeneous Gödel-type metric g for the functional $\mathcal{F}_t$ is established when the following set of equations is applied in Equation (2.3), given that $\|\varrho\|^2 = 12\omega^4 - 8\alpha\omega^2 + 2\alpha^2$.

$$\left\{\begin{array}{l} H\left((2t+1)\alpha^2 - 12\omega^2(t+1)\alpha + 10\omega^4(t+3)\right) = 0, \\ H^2\left((-2t-1)\alpha^2 + 12\omega^2(t+1)\alpha - 10\omega^4(t+3)\right) \\ \quad -D^2\left((2t+1)\alpha^2 - 8\omega^2(t+1)\alpha + 6\omega^4(t+3)\right) = 0, \\ \alpha^2(2t+1) - 4\omega^2\alpha(t+1) + 2\omega^4(t+3) = 0, \\ \alpha^2(2t+1) - 8\omega^2\alpha(t+1) + 6\omega^4(t+3) = 0, \\ \alpha^2(2t+1) - 12\omega^2\alpha(t+1) + 10\omega^4(t+3) = 0. \end{array}\right. \tag{3.4}$$

It is evident that the first two equations establish simultaneously, and it is imperative that we consider diverse solutions of the last three equations. The critical metrics for the functional $\mathcal{F}_t$ are summarized in the following theorem with utmost precision and accuracy.

Theorem 3.2. *A homogeneous Gödel-type space-time (M, g) is critical for the functional $\mathcal{F}_t$ if and only if one of the following cases occur*

I) $H = c_1, \quad D = c_2 + c_3 r$ *for any real value of* t.

II) $H = c_1 - c_2\omega r^2 - 2c_3\omega r, \quad D = c_2 r + c_3, \omega \neq 0,$ *for* $t = -3$,

III) $H = c_1, \begin{cases} D = c_2 e^{\sqrt{\alpha} r} + c_3 e^{-\sqrt{\alpha} r}, & \alpha > 0 \\ D = c_2 \cos(\sqrt{-\alpha} r) + c_3 \sin(\sqrt{-\alpha} r), & \alpha < 0 \end{cases}$ *, for* $t = -\frac{1}{2}$,

IV) $H = c_1 + c_2 e^{-2\omega r} - c_3 e^{2\omega r}, \quad D = c_2 e^{-2\omega r} + c_3 e^{2\omega r}, \quad \omega \neq 0,$ *for* $t = -\frac{1}{3}$,

where c_1, c_2, c_3 are arbitrary real constants.

Proof. Consider the following set of equations.

$$\begin{cases} eq_1 := \alpha^2(2t+1) - 4\omega^2\alpha(t+1) + 2\omega^4(t+3) = 0, \\ eq_2 := \alpha^2(2t+1) - 8\omega^2\alpha(t+1) + 6\omega^4(t+3) = 0, \\ eq_3 := \alpha^2(2t+1) - 12\omega^2\alpha(t+1) + 10\omega^4(t+3) = 0. \end{cases}$$

If $\alpha = 0$, then the above equations give $\omega^4(t+3) = 0$. Consider the following cases.

- If $\omega = 0$, then all of the equations vanish for any value of t, and the equation (2.5) gives $H' = D'' = 0$. The first statement is deduced by regular integration.
- If $\omega \neq 0$, then $t = -3$ and the equation (2.5) gives $H' = -2\omega D, D'' = 0$. The second statement is obtained by direct calculations.

If $\alpha \neq 0$,

- If $\omega = 0$, then $\alpha^2(2t+1) = 0$, which gives $t = -\frac{1}{2}$. From equation (2.5), we get $H' = 0, D'' = \alpha D$. The third statement is deduced by noticing that $H = c_1$ and by determining whether $\alpha > 0$ or $\alpha < 0$.
- If $\omega \neq 0$, then $\frac{1}{4\omega^2}(eq_2 - eq_1) = \omega^2(t+3) - \alpha(t+1) = 0$. Thus, $t = \frac{3\omega^2 - \alpha}{\alpha - \omega^2}$ and then eq_3 gives $\alpha(\alpha - 4\omega^2) = 0$, which immediately concludes $\alpha = 4\omega^2$. So, $t = -\frac{1}{3}$ and the equation (2.5) gives $H' = -2\omega D, D'' = 4\omega^2 D$. The last statement is obtained by direct calculations.

□

It is important to note that the first statement corresponds strictly to the flat (Einstein) solutions and the equation (2.3) is valid for any value of t. In addition, the last statement clearly indicates a conformally flat solution, which is also Bach-flat. As extensively studied in the literature, Bach-flat metrics are critical for $\mathcal{F}_{-1/3}$. Therefore, we strongly conclude that in the case of homogeneous Gödel-type space-times, Bach-flatness inevitably leads to conformal flatness.

4. Conclusion

This paper has thoroughly examined the critical metrics for functionals defined according to quadratic curvature invariants. These functionals have been widely studied in both geometric and physical contexts. Our study focused on the crucial Gödel-type space-time, which is of paramount importance in physical research. We have successfully provided a complete solution for the problem of classifying critical metrics for the functionals $\mathcal{F}_t$ and $\mathcal{S}$ for the homogeneous Gödel-type space-times.

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Talks

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THE LATTICES OF MÖBIUS TRANSFORMATIONS

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Abstract. The purpose of the note is to explore some properties of Möbius transformations. We obtain conditions that lattices of composition series of a Möbius group with finite composition length be a slim semimodular lattice.

Key words and phrases: Möbius transformation; group Möbius; slim semimodular lattice

1. Introduction

The theory of Möbius Transformations is developed without any use of and only one reference to complex analysis. Möbius transformations, named in honor of German mathematician August Ferdinand Möbius (1790-1868). The Möbius transform, which originated in the work of Rota [10], was introduced to deal with problems in combinatorics and number theory. There are numerous fields and results intertwining with the theory of the Möbius transform. Notice that there are many known connections between Möbius transform properties and slim semimodular lattices (see [10, 3, 6, 11]).

Slim semimodular lattices were introduced by G. Gratzer and E. Knapp in 2007, and they have been intensively studied since then. By a slim lattice we mean a finite lattice such that the poset (partially ordered set) of its non-zero join-irreducible elements, contains no three-element antichain. Semimodular lattices have recently proved to be useful in strengthening a classical group theoretical result, namely, the Jordan-Holder theorem. G. Grätzer and J. B. Nation [4] proved that given two composition series of a group, there is a matching between their factors such that the corresponding factors are isomorphic for a very specific reason. Every slim distributive lattice is dually slim.

This motivates the main result of the present paper, which asserts that lattices of composition series of a Möbius group with finite composition length is a slim semimodular lattice. Also, it is shown that the only Möbius transformation with more than two fixed-points is the identity.

2. Main results

In his fundamental paper [10], Rota introduced the Möbius inversion formula for any locally finite partially ordered set.

A Möbius transformation, is a map:

$$z \mapsto \frac{az+b}{cz+d}; \quad \left(z \in \hat{\mathbb{C}}, \quad a,b,c,d \in \mathbb{C}, \quad ad-bc \neq 0\right)$$

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It is clear that a Möbius transformation is holomorphic except for $z = -\frac{d}{c}$. Now, we make the assignments $f(-\frac{d}{c}) = \infty$ and $f(\infty) = \frac{a}{c}$ if $c \neq 0$. If $c = 0$, we assign $f(\infty) = \infty$ [5, 7].

Möbius geometry provides a unifying framework for studying planar geometries. In particular, the transformation groups of hyperbolic and elliptic geometries in the sections that follow are subgroups of the group of Möbius transformations [9].

We recall from [9] that under composition, Möbius transformations form a non-commutative group identifiable with a quotient of subgroups of the multiplicative group of invertible 2-by-2 matrices and there is a natural relationship between Möbius group operations and matrix group operations. The map

$$\tau : GL(2, \mathbb{C}) \to M$$

be given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} z \mapsto \dfrac{az+b}{cz+d} \end{bmatrix}$$

is a group homomorphism. The kernel of τ is the group of nonzero scalar matrices.

$$ker\tau = \left\{ \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}, k \neq 0 \right\}$$

Also, the inverse of a Möbius transformation is a Möbius transformation, and the composition of two Möbius transformations is a Möbius transformation and we have

$$PGL(2, \mathbb{C}) \approx M \quad [8].$$

The Möbius functions $M(z) = \frac{az+b}{cz+d}$ is determined by the values of four constant parameters a, b, c, d and very other $M(z)$ has one or two Fixed-Points $f = M(f)$ and we will have the following theorem.

Theorem 2.1. *The only Möbius transformation with more than two fixed-points is the identity,*

A lattice is a poset P any pair of elements x, y have a g.l.b. or meet denote by $x \wedge y$, and a l.u.b. or join denote by $x \vee y$. We recall form [2] that a lattice L is (upper) semimodular if, for all $x, y \in L$, $x \wedge y \prec\Longrightarrow y \prec x \vee y$. A semimodular lattice L is finite by definition, whence it has $0 = 0_L$ and $1 = 1_L$. Let B be a set B and $\mathfrak{C}$ be a subset $\mathfrak{C}$ of the power set $Pow(B) = \{X : X \subseteq B\}$ of B. We denote by $X \prec Y$ that Y covers X in $\mathfrak{C}$, that is, $X \subset Y$ but there is no $Z \in \mathfrak{C}$ such that $X \subset Z \subset Y$. We refer to [2] and [4] for some recent study related to lattices.

Definition 2.2. *Let $\overline{H} : \{1\} = H_0 \lhd H_1 \lhd \ldots \lhd H_n = G$ and be composition series of a Möbius group G. Denote $\{H_i \cap K_j : i, j \in \{0, \ldots, n\}\}$ by $Lat(\overline{H}, \overline{K})$. Clearly, $Lat(\overline{H}, \overline{K}) = (Lat(\overline{H}, \overline{K}), \subseteq)$ is a lattice.*

Now, we can prove that the following Theorem.

Theorem 2.3. *If G is a Möbius group with finite composition length and for any composition series $\overline{H}$ and $\overline{K}$ of G, then $Lat(\overline{H}, \overline{K})$ is a slim semimodular lattice.*

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CONFORMAL TRANSFORMATION OF CURVATURES IN FINSLER GEOMETRY

AKBAR TAYEBI AND FAEZEH ESLAMI

Abstract. In this paper, we study the conformal transformation of some important and effective non-Riemannian curvatures in Finsler Geometry. We find the necessary and sufficient condition under which the conformal transformation preserves the Berwald curvature, mean Berwald curvature, Landsberg curvature and mean Landsberg curvature.

Key words and phrases: Berwald curvature; mean Berwald curvature; Landsberg curvature; mean Landsberg curvature.

1. Introduction

The theory of conformal transformation (or change) of Finsler metrics has been studied by many Finsler geometers [2, 3, 4, 5, 6, 9]. But, Knebelman is the first person that studied the conformal theory of general Finsler metrics in [4]. He gave a geometrical criterion according to two Finsler metrics $\mathbf{g}(x,y) = g_{ij}(x,y)dx^i dx^j$ and $\tilde{\mathbf{g}}(x,y) = \tilde{g}_{ij}(x,y)dx^i dx^j$ to be conformal; this reduces to the usual requirement that $g_{ij} = e^{\kappa}\tilde{g}_{ij}$. In [5], he proved that the mentioned condition implies that $\kappa = \kappa(x)$ is a function of position, merely. Indeed, two Finsler metric functions $F = F(x,y)$ and $\bar{F} = \bar{F}(x,y)$ as conformal if the length of an arbitrary vector in the one is proportional to the length in the other. The classical Weyl theorem states that the projective and conformal properties of a Finsler metric determine the metrics properties uniquely. Thus, the conformal properties of the class of Finsler metric deserve extra attention.

In Finsler geometry, there are several important non-Riemannian quantities: the Berwald curvature $\mathbf{B}$, the mean Berwald curvature $\mathbf{E}$ and the Landsberg curvature $\mathbf{L}$, the mean Landsberg curvature $\mathbf{J}$, the non-Riemannian curvature $\mathbf{H}$, etc. They all vanish for Riemannian metrics, hence they are said to be non-Riemannian. In order to understand the conformal Finsler geometry, one can consider the conformal transformation of these non-Riemannian quantities.

The geodesics of F are characterized locally by the equation

$$\frac{d^2x^i}{dt^2} + 2G^i(x, \frac{dx}{dt}) = 0,$$

where G^i are coefficients of a spray $\mathbf{G}$ defined on M denoted by $\mathbf{G} = \delta/\delta x^i y^i$. Taking three vertical derivation of geodesic coefficients of F give us the Berwald curvature $\mathbf{B}$. A Finsler metric F is called a Berwald metric if $\mathbf{B} = 0$. In this case, $G^i = \Gamma^i_{jk}(x)y^j y^k$ are quadratic in $y \in T_xM$ for any $x \in M$. Every Berwald metric is a Landsberg metric.

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Taking a trace of Berwald curvature yields mean Berwald curvature **E**. Taking a horizontal derivation of the mean of Berwald curvature **E** give us the H-curvature **H**. In the class of Weyl metrics, vanishing this quantity results that the Finsler metric is of constant flag curvature and this fact clarifies its geometric meaning [1, 7]. By the definition, if $\mathbf{E} = 0$ then $\mathbf{H} = 0$.

In this paper, we study the conformal transformation of some important and effective non-Riemannian curvatures in Finsler Geometry. We find the necessary and sufficient condition under which the conformal transformation preserves the Berwald curvature **B**, mean Berwald curvature **E**, Landsberg curvature **L**, mean Landsberg curvature **J**, and the non-Riemannian curvature **H**.

There are many connections in Finsler geometry. Throughout this paper, we set the Berwald connection on Finsler manifolds. The h- and v- covariant derivatives of a Finsler tensor field are denoted by " | " and ", " respectively.

2. Conformal Transformation of (Mean) Berwald Curvature

In this section, we find the necessary and sufficient condition under which the conformal transformation preserves the Berwald curvature **B** and mean Berwald curvature **E**. For this aim, we need the following.

Theorem 2.1 ([8]). *Let F and $\bar{F}$ be two Finsler metrics on a manifold M. Then*

$$\bar{G}^i = G^i + \frac{g^{ij}}{4}\Big\{(\bar{F}^2)_{|k,j}y^k - (\bar{F}^2)_{|j}\Big\}, \tag{2.1}$$

where G^i and $\bar{G}^i$ are the geodesic spray coefficients of F and $\bar{F}$, respectively, and "|" and "," denote the horizontal and vertical derivation with respect to the Berwald connection of F.

Now, we can study the conformal transformation of Berwald curvature. We prove the following.

Theorem 2.2. *Let F and $\bar{F}$ be two Finsler metrics on a manifold M. If $\bar{F}(x,y) = e^{\sigma}F(x,y)$, then the conformal transformation preserves the Berwald curvature if and only if the conformal factor $\sigma = \sigma(x)$ satisfies following equation.*

$$\begin{aligned}
&2C_{jkl}\sigma^i - 2g_{jk}\sigma^m C^i{}_{ml} + 4\sigma^p C^m_{pl}(y_jC^i{}_{mk} + y_kC^i{}_{mj})\\
&- 2\sigma^m(g_{jl}C^i{}_{mk} + g_{kl}C^i{}_{mj} + y_jC^i{}_{mk,l} + y_kC^i_{mj,l}) + 4y_l\sigma^pC^m_{pk}C^i{}_{mj}\\
&- 4F^2\sigma^sC^p_{sl}C^m_{pk}C^i{}_{mj} + 2F^2\sigma^p(C^m_{pk,l}C^i_{mj} + C^m{}_{pk}C^i{}_{mj,l}) - 2y_l\sigma^mC^i{}_{mj,k}\\
&+ 2F^2\sigma^sC^m_{sl}C^i{}_{mj,k} - F^2\sigma^mC^i{}_{mj,k,l} = 0.
\end{aligned} \tag{2.2}$$

In particular, if $\sigma(x) = constant$, then $\bar{\mathbf{B}} = \mathbf{B}$.

Theorem 2.3. *Let F and $\bar{F}$ be two Finsler metrics on a manifold M. If $\bar{F}(x,y) = e^{\sigma}F(x,y)$, then the conformal transformation preserves the mean Berwald curvature if and only if the conformal transformation is homothetic or the conformal factor $\sigma = \sigma(x)$ satisfies following equation.*

$$\begin{aligned}
&4I_p(y_iC^p{}_{kj} + y_jC^p{}_{ki}) + 2F^2(C^p{}_{kj}I_{p,i} + C^p{}_{ki}I_{p,j})\\
&- 2(y_iI_{k,j} + y_jI_{k,i} + g_{ij}I_k) + F^2(2I_sC^s{}_{ki,j} - I_{k,i,j} - 4I_sC^s{}_{pi}C^p{}_{kj}) = 0.
\end{aligned} \tag{2.3}$$

In particular, if $\sigma(x) = constant$, then $\bar{\mathbf{E}} = \mathbf{E}$.

3. Conformal Transformation of Mean (Landsberg) Curvature

In this section, we find the necessary and sufficient condition under which the conformal transformation preserves the Landsberg curvature $\mathbf{L}$ and mean Landsberg curvature $\mathbf{J}$.

Theorem 3.1. *Let F and $\bar{F}$ be two Finsler metrics on a manifold M. If $\bar{F}(x,y) = e^{\sigma}F(x,y)$, then the conformal transformation preserves the Landsberg curvature if and only if the conformal factor $\sigma = \sigma(x)$ satisfies following equation.*

$$\begin{aligned}\sigma_0 C_{jkl} + \sigma^s \Big[& y_j C_{skl} + y_k C_{sjl} + y_l C_{sjk} + F^2 C_{jkl,s} \\ & - F^2 (C_{mjl} C^m_{\ sk} + C_{mjk} C^m_{\ sl} + C_{mkl} C^m_{\ sj}) \Big] = 0.\end{aligned}$$

In particular, if $\sigma(x) = constant$, then $\bar{\mathbf{L}} = \mathbf{L}$.

Theorem 3.2. *Let F and $\bar{F}$ be two Finsler metrics on a manifold M. If $\bar{F}(x,y) = e^{\sigma}F(x,y)$, then the conformal transformation preserves the mean Landsberg curvature if and only if the conformal factor $\sigma = \sigma(x)$ satisfies following equation.*

$$\sigma_0 I_l + \sigma^s \Big[I_s y_l + F^2 (I_{l,s} + 2 C^{jk}_{\ \ s} C_{jkl} - C^k_{\ ml} C^m_{\ sk} - C^j_{\ ml} C^m_{\ sj} - I_m C^m_{\ sl}) \Big] = 0.$$

In particular, if $\sigma(x) = constant$, then $\bar{\mathbf{J}} = \mathbf{J}$.

4. Conformal Transformation of H-Curvature

In this section, we find the necessary and sufficient condition under which the conformal transformation preserves the H-Curvature $\mathbf{H}$.

Theorem 4.1. *Let F and $\bar{F}$ be two Finsler metrics on a manifold M. If $\bar{F}(x,y) = e^{\sigma}F(x,y)$, then the conformal transformation preserves the H-curvature if and only if the conformal factor $\sigma = \sigma(x)$ satisfies following equation.*

$$\begin{aligned}Q^m_{mij|s} y^s = \sigma^p \Big[& 2F^2 E_{ij,p} + 2(E_{ip} y_j + E_{jp} y_i) - 2F^2 (E_{im} C^m_{\ pj} + E_{jm} C^m_{\ pi}) \\ & - F^2 Q^m_{mij,p} - (Q^m_{mip} y_j + Q^m_{mjp} y_i) + F^2 (Q^m_{mis} C^s_{\ pj} + Q^m_{mjs} C^s_{\ pi}) \Big].\end{aligned}$$

In particular, if $\sigma(x) = constant$, then $\bar{\mathbf{H}} = \mathbf{H}$.

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ON SQUARE FINSLER METRICS

AKBAR TAYEBI AND FATEMEH BARATI

Abstract. In this paper, we remark some of the well-known curvature properties of square metric. Then, we find the necessary and sufficient condition under which a square metric is weakly stretch.

Key words and phrases: Square metric; stretch curvature; mean stretch curvature.

1. Introduction

The well-known Hilbert's Fourth Problem is to characterize the distance functions on an open subset in $\mathbb{R}^n$ such that straight lines are shortest paths. It turns out that there are lots of solutions to the problem. For example, in [4], Blaschke discusses 2-dimensional solutions to the problem. Then, Ambartzumian [2] and Alexander [1] independently give all 2-dimensional solutions. In [8], Pogorelov discusses smooth solutions in 3-dimensional case. Then, Szabó investigates several problems left by Pogorelov and constructs continuous solutions to the problem in high dimensions [14]. See [5] on related issue.

The Hilbert Fourth Problem in the smooth case is to characterize Finsler metrics on an open subset in $\mathbb{R}^n$ whose geodesics are straight lines. Such Finsler metrics are called projectively flat Finsler metrics or projective Finsler metrics. Hamel first characterizes projective Finsler metrics by a system of PDE's [6]. Then, Rapcsák extends Hamel's result to projectively equivalent Finsler metrics [9].

For an n-dimensional Finsler manifold (M,F), a global vector field $\mathbf{G}$ is induced by F on $TM_0 := TM - \{0\}$, which in a standard coordinates (x^i, y^i) for TM_0 is given by $\mathbf{G} = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$, where $G^i = G^i(x,y)$ are called spray coefficients and given by following

$$G^i = \frac{1}{4} g^{il} \frac{\partial^2 F^2}{\partial x^k \partial y^l} y^k - \frac{\partial F^2}{\partial x^l}. \tag{1.1}$$

$\mathbf{G}$ is called the spray associated to F. F is projectively flat if only if there exists scalar homogeneous function $P : T\mathcal{U} \to \mathbb{R}$ such that the its spray coefficients satisfy

$$G^i(x,y) = P(x,y) y^i. \tag{1.2}$$

In this case, $P = P(x,y)$ is called the projective factor.

In Finsler Geometry, there is an interesting class of projectively flat metrics on the unit ball $\mathbb{B}^n$ which is given by

$$F = \frac{(\sqrt{(1-|x|^2)|y|^2 + \langle x, y\rangle^2} + \langle x, y\rangle)^2}{(1-|x|^2)^2\sqrt{(1-|x|^2)|y|^2 + \langle x, y\rangle^2}}. \tag{1.3}$$

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This class of metrics is called square metrics which can be expressed as

$$F = \frac{(\alpha+\beta)^2}{\alpha}, \tag{1.4}$$

where

$$\alpha = \frac{\sqrt{(1-|x|^2)|y|^2 + \langle x, y\rangle^2}}{(1-|x|^2)^2}, \quad \beta = \frac{\langle x, y\rangle}{(1-|x|^2)^2}. \tag{1.5}$$

That α is a Riemannian metric and β is a 1-form with $\|\beta\|_\alpha < 1$. L. Berwald first constructed a special projectively flat square metric of zero flag curvature on the unit ball in $\mathbb{R}^n$ (see [3]).

Then the flag curvature of F is a function $\mathbf{K} = \mathbf{K}(P, y)$ of tangent planes $P \subset T_xM$ and directions $y \in P$. F is called of scalar curvature if the flag curvature $\mathbf{K} = \mathbf{K}(x, y)$ is a scalar function on the slit tangent bundle TM_0, for any $y \in T_xM$. Recently, Shen-Yildirim determine the local structure of all locally projectively flat square metrics of constant flag curvature. Later on, L.Zhou shows that a square metric of constant flag curvature must be locally projectively flat. In [13], Shen-Yang proved the following.

Theorem 1.1 ([13])**.** *Let $F = (\alpha+\beta)^2/\alpha$ be a square metric on a $(n \geq 3)$-dimensional manifold M, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is Riemannian and $\beta = b_i(x)y^i$ is a 1-form on M. Then F is of scalar flag curvature if and only if it is locally projectively flat.*

A Finsler metric $F = F(x, y)$ on a manifold M is said to be locally dually flat if at any point there is a coordinate system (x^i) in which the spray coefficients are in the following form

$$G^i = -\frac{1}{2} g^{ij} H_{y^j},$$

where $H = H(x, y)$ is a C^∞ scalar function on $TM_0 = TM \setminus \{0\}$ satisfying $H(x, \lambda y) = \lambda^3 H(x, y)$ for all $\lambda > 0$.

Theorem 1.2 ([7])**.** *Let $F = (\alpha+\beta)^2/\alpha$ be a square metric on an open subset $U \subseteq \mathbb{R}^n$ with $n \geq 3$. Then F is dually flat if and only if one of the following holds:*
(i) F is a dually flat Riemannian metric.
(ii) F is of Minkowski-type. Moreover, F can be expressed in the following form.

$$F = \frac{(|y| + \langle v, y\rangle)^2}{|y|}, \tag{1.6}$$

where $v \in \mathbb{R}^n$ is a non zero constant vector.

Let (M, F) be an n-dimensional manifold Finsler manifold. Then F is called an Einstein metric if its Ricci curvature *Ric* is isotropic,

$$\mathbf{Ric} = (n-1)\lambda F^2,$$

where $\lambda = \lambda(x)$ is a scalar function on M. In [10], Shen-Yu proved the following.

Theorem 1.3 ([10])**.** *Let $F = (\alpha+\beta)^2/\alpha$ be a square metric on a n-dimensional manifold M, Then F is an Einstein metric if and only if it is Ricci flat and*

$$\begin{aligned} {}^{\alpha}Ric &= k^2(1-b^2)^2 - [5(n-1) + 2(2n-5)b^2]\alpha^2 + 6(n-2)\beta^2, \\ b_{i|j} &= k(1-b^2)(1+2b^2)a_{ij} - 3b_ib_j. \end{aligned} \tag{1.7}$$

Then, they determined the local structure of Einstein square metrics as follows.

Theorem 1.4 ([10]). *Let $F=(\alpha+\beta)^2/\alpha$ be a square metric on a n-dimensional manifold M, Then the following are equivalent.*
(1)F is an Einstein metric.
(2) The Riemaninnain metric $\tilde{\alpha}:=(1-b^2)\alpha$ and the 1-form $\tilde{\beta}:=\sqrt{1-b^2}\beta$ satisfy

$$\tag{1.8} {}^{\tilde{\alpha}}Ric=-(n-1)k^2\tilde{\alpha},\quad \tilde{b}_{i|j}=k\sqrt{1+\tilde{b}^2}\tilde{a}_{ij},$$

where k is a constant number, $\tilde{b}=\|\tilde{\beta}\|_{\tilde{\alpha}}$ and $\tilde{b}_{i|j}$ is the covariant derivation of $\tilde{\beta}$ with respect to $\tilde{\alpha}$. In this case, F is given in the following form

$$\tag{1.9} F=\frac{(\sqrt{1+\tilde{b}^2}\tilde{\alpha}+\tilde{\beta})^2}{\tilde{\alpha}},$$

with $(1+\tilde{b}^2)(1-b^2)=1$.
(3) The Riemannian metric $\bar{\alpha}:=(1-b^2)^{\frac{3}{2}}\sqrt{\alpha^2-\beta^2}$ and the 1-form $\bar{\beta}:=(1-b^2)\beta$ satisfy $\bar{\alpha}_{Ric}=0$ and $\bar{b}_{i|j}=k\bar{a}_{ij}$ where k is a constant number, $\bar{b}=\|\bar{\beta}\|_{\bar{\alpha}}$ and $\bar{b}_{i|j}$ is the covariant derivation of $\bar{\beta}$ with respect $\bar{\alpha}$. In this case, F is given in the following form

$$\tag{1.10} F=\frac{(\sqrt{(1-\bar{b}^2)\bar{\alpha}^2+\bar{\beta}^2}+\bar{\beta})^2}{(1-\bar{b}^2)^2\sqrt{(1-\bar{b}^2)\bar{\alpha}^2+\bar{\beta}^2}},$$

with $\bar{b}=b$.

Also, they provide a new description for square metrics with constant flag curvature.

Theorem 1.5 ([10]). *The Finsler metric $F=(\alpha+\beta)^2/\alpha$ is of constant flag curvature if and only if under the expression (1.10) of F, $\bar{\alpha}$ is locally Euclidean, $\bar{\beta}$ is closed and s homothety with respect to $\bar{\alpha}$. In a suitable local coordinate, F can be expressed by*

$$\tag{1.11} F=\frac{(\sqrt{(1-|\bar{x}|^2)|y|^2+\langle\bar{x},y\rangle^2}+\langle\bar{x},y\rangle)^2}{(1-|\bar{x}|^2)^2\sqrt{(1-|\bar{x}|^2)|y|^2+\langle\bar{x},y\rangle^2}},$$

where $\bar{x}:=cx+a$ for some constant number c and constant vector a. In particular, F must be locally projectively flat with zero flag curvature.

For $y\in T_xM$, define the Landsberg curvature $\mathbf{L}_y:T_xM\otimes T_xM\otimes T_xM\longrightarrow\mathbb{R}$ by

$$\mathbf{L}_y(u,v,w):=\frac{-1}{2}g_y(\mathbf{B}_y(u,v,w),y).$$

A Finsler metric F is called a Landsberg metric if $\mathbf{L}_y=0$.

For $y\in T_xM$, define $J_y:T_xM\longrightarrow\mathbb{R}$ by $J_y(u):=J_i(y)u^i$. J is called the mean Landsberg curvature. A Finsler metric F is called a weakly Landsberg metric if $J_y=0$. For $y\in T_xM$, define the stretch curvature $\mathbf{\Sigma}_y:T_xM\otimes T_xM\otimes T_xM\otimes T_xM\longrightarrow\mathbb{R}$, by $\mathbf{\Sigma}_y(u,v,w,z):=\Sigma_{ijkl}(y)u^iv^jw^kz^l$, where

$$\tag{1.12} \Sigma_{ijkl}:=2(L_{ijk|l}-L_{ijl|k}),$$

and $'|'$ denotes the horizontal derivation with respect to the Berwald connection of F. A Finsler metric F is said to be a stretch metric if $\mathbf{\Sigma}=0$. Also, one can define Mean stretch curvature $\overline{\Sigma}_y:T_xM\longrightarrow\mathbb{R}$ by $\overline{\Sigma}_y(u,v):=\overline{\Sigma}_{ij}(y)u^iv^j$, where

$$\tag{1.13} \overline{\Sigma}_{ij}:=2(J_{i|j}-J_{j|i}).$$

A Finsler metric F is said to be weakly stretch metric if $\overline{\Sigma} = 0$. It is easy to see that every Landsberg metric or stretch metric is a weakly stretch metric.

For an (α, β)-metric, let us define $b_{i|j}$ by $b_{i|j}\theta^j := db_i - b_j\theta_i^j$, where $\theta^i := dx^i$ and $\theta_i^j := \Gamma^j_{ik}dx^k$ denote the Levi-Civita connection form of α. Let

$$r_{ij} := \frac{1}{2}(b_{i|j} + b_{j|i}), \quad r_{00} := r_{ij}y^iy^j.$$

In this paper, we prove the following.

Theorem 1.6. *The Finsler metric $F = (\alpha+\beta)^2/\alpha$ is weakly stretch if and only if the following relationship holds:*

$$Ar_{00}^2 + Br_{00} + C = 0 \tag{1.14}$$

Above relationship is equivalent to the following two equations:

$$A_1r_{00}^2 + B_1r_{00} + C_1 = 0, \tag{1.15}$$

$$A_2r_{00}^2 + B_2r_{00} + C_2 = 0, \tag{1.16}$$

where $A_1, A_2, B_1, B_2, C_1, C_2$ are functions in terms of b^2 and s.

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THE RIEMANN CURVATURE OF A SPECIAL CLASS OF FINSLER METRIC

NASRIN SADEGHZADEH AND NAJMEH SAJJADI

Abstract. In this paper the Riemann curvature of a Special class of generalized (α, β)-metrics are considered. In particular the conditions for begin R-quadratic and PR-quadratic types are investigated.

Key words and phrases: General (α, β)-metrics; R-quadratic ; PR-quadratic.

1. Introduction

Finsler geometry is a appropriate extension of Riemannian geometry. According to Finsler geometry it was already discussed by Riemann in his lecture in 1854 [12]. Afterward, the systematic study of these spaces appeared in the thesis of Finsler in 1918 [8].

(α, β)-metrics form a special class of Finsler metrics to some extent, because they are "computable". The significance of (α, β)-metrics was firstly suggested by M. Matsumoto in 1972 as a direct generalization of Randers metrics [11]. In this research we are going to concentrate on an important class of Finsler metrics called general (α, β)-metrics, which are given as

$$F = \alpha\phi(b^2, s),$$

where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on M, respectively. $b^2 = b^i b_i$, $s = \frac{\beta}{\alpha}$ and ϕ is a smooth function. A Finsler metric is said to be R-quadratic if its Riemann curvature is quadratic [7]. R-quadratic metrics were first introduced by Báscó and Matsumoto [3].

For a Finsler space (M, F), the Riemann curvature is a family of linear transformations

$$\mathbf{R}_y : T_xM \to T_xM,$$

where $y \in T_xM$, with homogeneity $\mathbf{R}_{\lambda y} = \lambda^2 \mathbf{R}_y$, $\forall \lambda > 0$. The Finsler metric (M, F) is *R-quadratic* if $\mathbf{R}_y$ is quadratic in $y \in T_xM$. Here a special class of general (α, β)-metric of R-quadratic type is considered. In general, it is difficult to find the Riemann curvature tensor for general (α, β)-metrics. Then we consider the metrics under the following assumption

$$ {}^{\alpha}R^i{}_k = \mu(\alpha^2 \delta^i{}_k - y_k y^i), \quad b_{i|k} = c(x)a_{ik}, \tag{1.1}$$

where ${}^{\alpha}R^i{}_k$ denotes the Riemann curvature of the Riemannian metric α and μ is the Ricci constant.

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The concept of Projective Ricci curvature $PRic$ for a Finsler metric F is defined by Z. Shen [13], as follow

$$PRic = Ric + (n-1)\left(\frac{S_{|m}y^m}{n+1} + \frac{S^2}{(n+1)^2}\right).$$

In fact, for a Finsler metric (M, F), the Riemann curvature of a projective spray is called Projective Riemann curvature. The Projective Ricci curvature is defined as the Ricci curvature of the projective spray. The Projective Ricci curvature of Finsler metric on a manifold M is projective invariant with respect to a fixed volume form on M.

A Finsler metric (M, F) is called PR-quadratic if its Projective Riemann curvature PR_y is quadratic in $y \in T_xM$. In this research, after considering a special class of generalized (α, β)-metrics of R-quadratic type, the conditions for being of PR-quadratic type are studied.

2. Main results

Theorem 2.1. *Let (M, F) be a general (α, β)-metric satisfying (1.1). Then F is of R-quadratic type if and only if*

$$R^i_k = R_2\theta^i_{pkq}(x)y^py^q, \tag{2.1}$$

where

$$\theta_j{}^i{}_{kl}(x) = (a_{jl}b^2 - b_jb_l)\delta^i{}_k - (a_{jk}b^2 - b_jb_k)\delta^i{}_l + (b_la_{jk} - a_{jl}b_k)b^i,$$

and $R_2 = R_2(r)$ as

$$R_2 = -\mu(2\chi - s\chi_s) + c^2[2(2\psi_{b^2} - s\psi_{b^2s}) - \chi_{ss} + 2\chi(2\chi - s\chi_s) + (b^2 - s^2)(2\chi\chi_{ss} - \chi_s^2).$$

Theorem 2.2. *A general (α, β)-metric (M, F), satisfying (1.1) is of PR-quadratic type if and only if*

$$R^i{}_k = \alpha^2\left\{R_1{}^\alpha h^i{}_k + R_3s_{.k}y^i\right\}, \tag{2.2}$$

where R_1 and R_3 satisfying the following equations

$$\alpha^2R_1 = E + \mu_{pq}(x)y^py^q, \quad -\frac{3}{n+1}\chi_k = \alpha^2(2R_3 + (R_1)_s)s_{.k},$$

and χ_k is the χ-curvature of F and $E = \frac{S^2}{(n+1)^2} + \frac{s_{|0}}{n+1}$.

Here R_1 and R_2 are the same as stated in [14].

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SOME NEW NON-RIEMANNIAN QUANTITIES IN FINSLER GEOMETRY

NASRIN SADEGHZADEH

Abstract. In this paper, some new Non-Riemannian quantities and then new classes of Finsler metrics are considered. All these Finsler metrics are contained in the projective invariant class of Finsler metrics, generalized Douglas Weyl (GDW)-metrics. In fact, some new sub-classes of GDW-metrics are constructed and considered as the explicit Finsler metrics. Many illustrative and interesting examples are presented.

Key words and phrases: Douglas metrics; $\bar{D}$-metrics; R-quadratic Finsler metrics; PR-quadratic Finsler metrics; GDW-metrics; $GD\widetilde{W}$-metrics.

1. Introduction

Two regular metrics on a manifold M are called projectively related if they have the same geodesics as the point sets. In Physics, a geodesic represents the equation of motion that determines the phenomena of the space. A geodesic curve $c(t)$ in a Finsler space (M, F), is defined by the second order system of differential equations

$$\frac{d^2c^i}{dt^2} + 2G^i(c(t), \dot{c}(t)) = 0,$$

where $G^i(y)$ are local functions on TM given by

$$G^i(y) := \frac{1}{4}g^{il}(y)\{\frac{\partial^2 F^2}{\partial x^k \partial y^l}y^k - \frac{\partial F^2}{\partial x^l}\},$$

for $y \in T_xM$.

Projective Finsler geometry studies equivalent Finsler metrics on the same manifold with the same geodesics as points. There are well-known projective invariants of Finsler metrics namely Douglas curvature, Weyl curvature [2], generalized Douglas-Weyl curvature [3].

A C-projective invariant H-curvature introduced by Akbar-Zadeh [1], too. C-projective Weyl curvature ($\widetilde{W}$-curvature), the new C-projectively invariant quantity which characterizes Finsler metrics of constant flag curvature is presented in [5].

The Finsler metrics satisfying,

$$D_j{}^i{}_{kl|m}y^m = T_{jkl}y^i,$$

for some tensor T_{jkl}, where $D_j{}^i{}_{kl|m}$ denotes the horizontal covariant derivatives of $D_j{}^i{}_{kl}$ with respect to the Berwald connection of F, are called GDW-metrics [3]. Although, all Douglas metrics are of GDW type, there are many GDW Finsler metrics which are not of Douglas type. The following example presents a GDW-metric which is not of Douglas type.

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Example 1.1 ([4]). *Put $\Omega = \{(x, y, z) \in R^3 | x^2 + y^2 + z^2 < 1\}$, $p = (x, y, z) \in \Omega$ and $y = (u, v, w) \in T_p\Omega$. Define the Randers metric $F = \alpha + \beta$ by*

$$\alpha = \frac{\sqrt{(-yu + xv)^2 + (u^2 + v^2 + w^2)(1 - x^2 - y^2)}}{1 - x^2 - y^2}, \quad \beta = \frac{-yu + xv}{1 - x^2 - y^2}.$$

The above Randers metric has vanishing flag curvature $K = 0$ and S-curvature $S = 0$. F has zero Weyl curvature then F is of GDW metric. But β is not closed then F is not of Douglas type.

On the other hands, the class of Douglas metrics contains all Riemannian metrics and the locally projectively flat Finsler metrics. But, there are many Douglas metrics which are not Riemannian. There are many Douglas metrics which are not locally projectively flat, too.
The following example presents a Douglas metric which is not locally projectively flat.

Example 1.2 ([8]). *Define α and β by*

$$\widetilde{\alpha} = \eta^{\frac{m}{1-m}}\alpha, \quad \widetilde{\beta} = \eta^{-1}\beta,$$

for some $\eta = \eta(x)$ and $\widetilde{\beta}$ is parallel with respect to $\widetilde{\alpha}$ where $\widetilde{\alpha}$ and $\widetilde{\beta}$

$$\widetilde{\alpha} = \sqrt{\frac{|y|^2}{|u|^2}}, \quad \widetilde{\beta} = \frac{< x, y >}{|u|^2},$$

and $u = (u^1(x), \dots, u^n(x))$ is a vector satisfying the following

$$u^i = -2(\lambda + t < f, x >)x^i + t|x|^2 f^i + f^i,$$

where t is a constant and f is a constant vector satisfying $tf \neq 0$ and $\lambda^2 + t|f|^2 \neq 0$. Then the m-Kropina metric $F = \alpha^m \beta^{1-m}$ is Douglasian but not locally projectively flat, where $m \neq 0, 1$.

Based on Douglas curvature, a new class of Finsler metrics so called $\bar{D}$-metrics is introduced which includes all the Douglas metrics.
A Finsler metric F is called $\bar{D}$-metric if $D_j{}^i{}_{kl|m} - D_j{}^i{}_{km|l} = 0$ or equivalently $D_j{}^i{}_{kl|m} y^m = 0$.
Clearly, the class of $\bar{D}$-metrics contains all Douglas metrics but there are many $\bar{D}$-metrics which are not Douglas. It means that

$$\begin{aligned} \{\text{Locally Projectively Flat}\} &\subsetneqq \{\text{Douglas metrics}\} \\ &\subsetneqq \{\bar{D} - \text{metrics}\} \\ &\subsetneqq \{\text{GDW-metrics}\}. \end{aligned}$$

There are other interesting classes of Finsler metrics which are the subset of the class of GDW-metrics. R-quadratic and PR-quadratic Finsler metrics are some great examples of them. The Riemann curvature is one of the important quantities, in Finsler geometry. For a Finsler space (M, F), the Riemann curvature is a family of linear transformations

$$\mathbf{R}_y : T_x M \to T_x M,$$

where $y \in T_x M$, with homogeneity $\mathbf{R}_{\lambda y} = \lambda^2 \mathbf{R}_y$, for every $\lambda > 0$. R-quadratic Finsler spaces form a rich class of Finsler spaces which were introduced by Z. Shen and could be considered as a generalization of Berwald metrics. A Finsler metric (M, F) is called R-quadratic if its Riemann curvature R_y is quadratic in $y \in T_x M$.

In [6], it is proved that every R-quadratic Finsler metric is a GDW-metric.
However, the tensors which contain both Ricci curvature $Ric = Ric(x, y)$ and S-curvature $S = S(x, y)$ are more applicable [7]. Here, the Ricci curvature is defined as the trace of the Riemann curvature. For a Finsler metric (M, F), the Riemann curvature of a projective spray is called Projective Riemann curvature (PR_y). The Projective Ricci curvature is defined as the Ricci curvature of the projective spray, too.
The concept of Projective Ricci curvature $PRic$ for a Finsler metric F is defined by Z. Shen [7], as follows

$$PRic = Ric + (n-1)\big(\frac{S_{|m}y^m}{n+1} + \frac{S^2}{(n+1)^2}\big).$$

The Projective Ricci curvature of Finsler metrics on a manifold M is projective invariant with respect to a fixed volume form on M.
A Finsler metric (M, F) is called PR-quadratic if its Projective Riemann curvature PR_y is quadratic in $y \in T_xM$.
It is proved that this class of Finsler metrics contains the class of Douglas metrics $(D(M))$ and belongs to the class of GDW-metrics $(GDW(M))$.
This paper also defines a new quantity in Finsler geometry, so-called generalized Berwald projective Weyl $(GB\widetilde{W})$ curvature, which is a C-projective invariant.
For manifold M, let GB$\widetilde{W}(M)$ denotes the class of all Finsler metrics satisfying

$$B_j{}^i{}_{kl} = \beta_j{}^i{}_{kl} + b_{jkl}y^i,$$

for some tensors b_{jkl} and $\beta_j{}^i{}_{kl}$; where $\beta_j{}^i{}_{kl|m}y^m = 0$.
A natural question that could be raised is: "How large is GB$\widetilde{W}(M)$ and what kind of interesting metrics does it have?"
It is clear that all Berwald metrics belong to this class. However, the Berwald metrics are not C-projective invariants. It is shown that the class of $GB\widetilde{W}$-metrics is the proper subset of the class of GDW-metrics.

2. Main results

In this section, the main results of this research are presented.
For convenience, we use the following notations. For Finsler manifold (M, F),
$D(M)$ denotes the class of all Douglas metrics,
$B(M)$ denotes the class of all Berwald metrics,
$\bar{D}(M)$ denotes the class of all $\bar{D}$-metrics,
$GB\widetilde{W}(M)$ denotes the class of all Generalized Berwald Projective Weyl $(GB\widetilde{W})$-metrics,
$GDW(M)$ denotes the class of all Generalized Douglas Weyl (GDW)-metrics,
$PRq(M)$ denotes the class of all PR-quadratic Finsler metrics,
$Rq(M)$ denotes the class of all R-quadratic Finsler metrics,
on the manifold M.

Definition 2.1. *A Finsler metric F is called $\bar{D}$-metric if $D_j{}^i{}_{kl|m} - D_j{}^i{}_{km|l} = 0$ or equivalently $D_j{}^i{}_{kl|m}y^m = 0$*

Theorem 2.2. *For a Finsler manifold (M, F), $D(M)$ is a proper subset of $\bar{D}(M)$ and $\bar{D}(M)$ is a proper subset of $GDW(M)$.*

Definition 2.3. *A Finsler metric (M, F) is called PR-quadratic if its Projective Riemann curvature PR_y is quadratic in $y \in T_xM$.*

Theorem 2.4. *For a Finsler manifold (M, F), $D(M)$ is a proper subset of $PRq(M)$ and $PRq(M)$ is a proper subset of $GDW(M)$.*

Definition 2.5. *A Finsler metric F is called $GB\widetilde{W}$ if its Berwald curvature satisfies*

$$B_j{}^i{}_{kl} = \beta_j{}^i{}_{kl} + b_{jkl}y^i,$$

for some tensors b_{jkl} and $\beta_j{}^i{}_{kl}$; where $\beta_j{}^i{}_{kl|m}y^m = 0$.

Theorem 2.6. *For a Finsler manifold (M, F), $Rq(M)$ (and then $B(M)$) is a proper subset of $GB\widetilde{W}(M)$ and $GB\widetilde{W}(M)$ is a proper subset of $GDW(M)$.*

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ON A CLASS OF DOUGLAS METRICS WITH RELATIVELY ISOTROPIC LANDSBERG CURVATURE

MEHRAN GABRANI AND BAHMAN REZAEI

Abstract. In this paper, we classify Finsler warped product metrics with relatively isotropic Landsberg curvature. This study then proves that Douglas type Finsler warped product metrics with relatively isotropic Landsberg curvature must be either Berwald metrics or Randers metrics.

Key words and phrases: Douglas metric; Finsler warped product metric; Isotropic Landsberg curvature.

1. Introduction

Very recently, Chen, Shen, and Zhao introduced a new class of Finsler metric that is an extension of the Finsler geometry of the concept of the warped product structure on an n-dimensional manifold $M := I \times \breve{M}$ where I is an interval of $\mathbb{R}$ and $\breve{M}$ is an $(n-1)$-dimensional manifold equipped with a Riemannian metric, [1]. In fact, it is considered in the following form:

$$F(u,v) = \breve{\alpha}(\breve{u},\breve{v})\phi\left(u^1, \frac{v^1}{\breve{\alpha}(\breve{u},\breve{v})}\right), \tag{1.1}$$

where $u = (u^1, \breve{u}), v = v^1\frac{\partial}{\partial u^1} + \breve{v}$ and ϕ is a suitable function defined on a domain of $\mathbb{R}^2$. Throughout this paper, the index conventions are as follows:

$$1 \leq A \leq B \leq \ldots \leq n, \qquad 2 \leq i \leq j \leq \ldots \leq n.$$

The class of spherically symmetric Finsler metrics can be regarded as Finsler warped product metrics. The flag curvature and Ricci curvature of Finsler warped product metrics are obtained by Chen-Shen-Zhao, [1]. In [4], Gabrani, Rezaei, and Sevim characterized Finsler warped product metrics with isotropic mean Berwald curvature. Moreover, they studied and classified the Landsberg Finsler warped product metrics [3]. Moreover, Gabrani, Rezaei, and Sevim studied the volume form dV on an n-dimensional manifold, which admits the Finsler warped product metrics, to introduce and classify the S-curvature of Finsler warped product metrics [2].

We have the following result for Finsler warped product metrics.

Theorem 1.1. *Let $F(u,v) = \breve{\alpha}(\breve{u},\breve{v})\phi\left(u^1, \frac{v^1}{\breve{\alpha}(\breve{u},\breve{v})}\right)$ be a non-Riemannian Douglas Finsler warped product metric on an n-dimensional manifold M ($n \geq 3$), where $u = (u^1, \breve{u})$, $v = v^1\dfrac{\partial}{\partial u^1} + \breve{v}$ and $\phi \in C^\infty$. If F has relatively isotropic Landsberg curvature, then either*

1. *F is Berwaldian or*

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Speaker: Mehran Gabrani.

2. *F is a Randers metric that is of the following form*

$$F = \frac{2cv^1 + \sqrt{(4c^2 + \sigma a_1(r))\,(v^1)^2 - \sigma a_4(r)\breve{\alpha}^2}}{\sigma}, \tag{1.2}$$

where $c = c(r) \neq 0, \sigma = \sigma(u) \neq 0, a_1 = a_1(r)$ and $a_4 = a_4(r)$ are differentiable functions.

Then, we give the following examples to see the geometric view of the theorems.

Example 1.2. *Take $c(r) = r, \sigma = \lambda, a_1(r) = [p(r) + \frac{1}{2}q(r)]r^2$ and $a_4(r) = -\frac{1}{2}q(r)r^2$ in (1.2); then, we have the following example obtained by Zhao Yang and Xiaoling Zhang:*

$$F = \frac{2rv^1 + \sqrt{\left(4c^2 + \lambda p(r) + \frac{1}{2}\lambda q(r)\right) r^2 (v^1)^2 + \frac{1}{2}\lambda q(r) r^2 \breve{\alpha}^2}}{\lambda}.$$

Then, F is a new warped product Finsler metric with relatively isotropic Landsberg curvature, [5].

2. Preliminaries

In this section, we briefly introduce some geometric quantities and definitions in Finsler geometry to prove the main theorems in this paper.

Let M be an n-dimensional manifold. It is known that a Finsler metric is a nonnegative function $F(u, v)$ on TM, which has the following properties.

(a) $F(u, v)$ is C^∞ on $TM \backslash \{0\}$;

(b) the restriction $F_u := F_{|T_u M}$ is a Minkowski function on $T_u M$ for all $u \in M$.

Assume that F is a Finsler metric on an n-dimensional manifold M. In local coordinates $u^1, \ldots, u^n$ and $v = v^A \dfrac{\partial}{\partial v^A}$, $\mathbf{G} = v^A \dfrac{\partial}{\partial u^A} - 2G^A \dfrac{\partial}{\partial v^A}$ is a spray induced by F. The spray coefficients G^A are locally expressed as follows:

$$G^A := \frac{1}{4} g^{AB} \{[F^2]_{u^C v^B} v^C - [F^2]_{u^B}\},$$

where $g_{AB}(u, v) = \left[\frac{1}{2}F^2\right]_{v^A v^B}$ and $(g^{AB}) = (g_{AB})^{-1}$.

The Cartan torsion of a Finsler metric is given by

$$C_{IJK} = \frac{1}{2} \frac{\partial g_{IJ}}{\partial v^K}.$$

Then, the mean Cartan torsion $\mathbf{I} = I_M du^M$ is defined by

$$I_M := g^{JK} C_{MJK}. \tag{2.1}$$

Moreover, the Landsberg metrics are defined by

$$L_{CDE} := -\frac{1}{2} F F_{v^A} \frac{\partial^3 G^A}{\partial v^C \partial v^D \partial v^E} = 0. \tag{2.2}$$

It is known that the mean Landsberg curvature $\mathbf{J} = J_M du^M$ is defined by

$$J_M := g^{JK} L_{MJK}. \tag{2.3}$$

A Finsler metric is considered to be of relatively isotropic Landsberg curvature if $\mathbf{L} + cF\mathbf{C} = 0$, where $c = c(u)$ is a scalar function on the manifold. It is known that a Finsler metric F is of relatively isotropic mean Landsberg curvature if and only if $\mathbf{J} + cF\mathbf{I} = 0$.

Proposition 2.1. *Let $F(u,v) = \breve{\alpha}(\breve{u},\breve{v})\phi\Big(u^1, \frac{v^1}{\breve{\alpha}(\breve{u},\breve{v})}\Big)$ be a non-Riemannian Finsler warped product metric on an n-dimensional manifold M where $u = (u^1, \breve{u})$, $v = v^1\dfrac{\partial}{\partial u^1} + \breve{v}$ and $\phi \in C^\infty$. Then, F has relatively isotropic Landsberg curvature if and only if there exist functions $c = c(r)$ and $a_i(r), i \in \{1,2,3,4\}$, such that*

$$\Psi = c\phi + a_1(r)s + a_2(r), \tag{2.4}$$

$$A = \frac{1}{2}s^2 a_3(r) - a_2(r)s + a_4(r). \tag{2.5}$$

3. Proof of main theorems

Now, we will classify the Douglas Finsler warped product metrics with relatively isotropic Landsberg curvature.

Proof of Theorem 1.1: Assume that F has relatively isotropic Landsberg curvature. Then, (2.4) and (2.5) hold. Assume that F is of a Douglas type Finsler warped product metric. Then, we obtain $a_2(r) = 0$. Thus,

$$\Psi = c\phi + a_1(r)s, \tag{3.1}$$

$$A = \frac{1}{2}s^2 a_3(r) + a_4(r). \tag{3.2}$$

We consider the following cases:

Case 1. $c = 0$, then $A = \frac{1}{2}s^2 a_3(r) + a_4(r)$ and $\Psi = a_1(r)s$, which means that F is Berwaldian.

Case 2. $c \neq 0$: In this case, by the proof of Proposition 4 in [2], we can see that if $F = \breve{\alpha}\phi(r,s)$ is a Finsler warped product metric where ϕ satisfies (3.1) and (3.2), then either F is a Randers warped product (Riemann warped product included), which can be formulated by

$$F = \frac{2cv^1 + \sqrt{\left(4c^2 + \sigma a_1(r)\right)(v^1)^2 - \sigma a_4(r)\breve{\alpha}^2}}{\sigma},$$

where $c = c(r) \neq 0, \sigma = \sigma(u) \neq 0, a_1 = a_1(r)$ and $a_4 = a_4(r)$ are differentiable functions; or F is a singular Kropina warped product metric, which we omit.

□

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ON LANDSBERG QUINTIC FINSLER METRICS

JILA MAJIDI, AKBAR TAYEBI, AND ALI HAJI-BADALI

Abstract. In this paper, we study a class of 5-th root (α, β)-metrics. We show that the Landsberg 5-th root (α, β)-metric has vanishing S-curvature.

Key words and phrases: Landsberg curvature; S-Curvature; m-th root; (α, β)-metric.

1. Introduction

Let $F = F(x, y)$ be a Finsler metric on tangent bundle TM defined as $F = \sqrt[m]{A}$, where $A := a_{i_1 \dots i_m}(x) y^{i_1} y^{i_2} \dots y^{i_m}$ and $a_{i_1 \dots i_m}$ are symmetric in all its indices. Then, F is called an m-th root Finsler metric on the manifold M. The class of m-th root Finsler metrics has been developed by Shimada in [3], and applied to biology as an ecological metric by Antonelli in [1].
The fifth root metrics $F = \sqrt[5]{a_{ijklp}(x) y^i y^j y^k y^l y^p}$ are called the quintic metrics. In order to understand the structure of quintic root metrics, one can study the non-Riemannian curvatures of these metrics. Among these quantities, the mean Landsberg curvature $\mathbf{J}$ and the S-curvature $\mathbf{S}$ have important and deep relation with each other. Let us give a brief explanation of their relations. The distortion $\tau = \tau(x, y)$ is a non-Riemannian quantity that is determined by the Busemann-Hausdorff volume form. The vertical and horizontal derivations of distortion τ on each tangent space give rise to the mean Cartan torsion $\mathbf{I} := \tau_{y^s} dx^s$ and S-curvature $\mathbf{S} = \tau_{|t} y^t$.

Theorem 1.1. *Let $F = \sqrt[5]{c_1 \alpha^4 \beta + c_2 \alpha^2 \beta^3 + c_3 \beta^5}$ be an (α, β)-metric on a manifold M. Then $\mathbf{L} = 0$ if it is a Berwald metric.*

2. Preliminaries

Let $F = \alpha\phi(s)$, $s = \beta/\alpha$, be an (α, β)-metric, where $\phi = \phi(s)$ is a C^∞ on $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{jt}(x) y^j y^t}$ is a Riemannian metric and $\beta = b_j(x) y^j$ is a 1-form on a manifold M. For an (α, β)-metric, let us define $b_{j;k}$ by $b_{j;k}\theta^k := db_j - b_k \theta_j^k$, where $\theta^j := dx^j$ and $\theta_j^k := \Gamma^k_{js} dx^s$ denote the Levi-Civita connection form of α. Let

$$r_{it} := \frac{1}{2}(b_{i;t} + b_{t;i}), \quad s_{it} := \frac{1}{2}(b_{i;t} - b_{t;i}), \quad r_{i0} := r_{it} y^t, \quad r_{00} := r_{it} y^i y^t, \quad r_t := b^i r_{it},$$
$$s_{i0} := s_{it} y^t, \quad s_t := b^i s_{it}, \quad s^i{}_t = a^{is} s_{st}, \quad s^i{}_0 = s^i{}_t y^t, \quad r_0 := r_t y^t, \quad s_0 := s_t y^t,$$

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where $a^{it} = (a_{it})^{-1}$ and $b^i := a^{it}b_t$. Put

$$Q := \frac{\phi'}{\phi - s\phi'}, \quad \Theta := \frac{\phi\phi' - s(\phi'\phi' + \phi\phi'')}{2\phi((\phi - s\phi') + (B - s^2)\phi''}, \tag{2.1}$$
$$\Psi := \frac{\phi''}{2[(\phi - s\phi') + (B - s^2)\phi'']},$$

where $B := ||\beta||^2_\alpha$. Let $G^t = G^t(x, y)$ and $G^t_\alpha = G^t_\alpha(x, y)$ denote the coefficients of F and α, respectively, in the same coordinate system. By definition, we have

$$G^t = G^t_\alpha + \alpha Q {s^t}_0 + (r_{00} - 2Q\alpha s_0)(\alpha^{-1}\Theta y^t + \Psi b^t), \tag{2.2}$$

where

$$P := \Big[-2Q\alpha s_0 + r_{00}\Big]\Theta\alpha^{-1}, \quad Q^t := \Psi\Big[r_{00} - 2\alpha Q s_0\Big]b^t + \alpha Q {s^t}_0.$$

Clearly, if β is parallel with respect to α, that is $r_{ij} = 0$ and $s_{ij} = 0$, then $P = 0$ and $Q^i = 0$. In this case, $G^i = G^i_\alpha$ are quadratic in y. In this case, F is a Berwald metric. Put

$$\Phi : = (sQ' - Q)\{n\Delta + sQ + 1\} - (B - s^2)(sQ + 1)Q''.$$

By a direct computation, we can obtain a formula for the mean Cartan torsion of (α, β)- metrics as follows

$$I_j = -\frac{(\phi - s\phi')\Phi}{2\Delta\phi\alpha^2}(\alpha b_j - sy_j). \tag{2.3}$$

Thus $\mathbf{I} = 0$ if and only if $\Phi = 0$.

3. Proof of Theorem 1.1

The Landsberg tensor $\mathbf{L} = L_{ijk}(x, y)dx^i \otimes dx^j \otimes dx^k$ is defined by

$$L_{ijk} := -\frac{1}{2}FF_{y^m}[G^m]_{y^iy^jy^k}.$$

In [2], Shen obtained the following form of the expression for L_{jkl} .

Lemma 3.1. *Let*

$$L_{jkl} = -\frac{\rho}{6\alpha^5}\Big\{h_jh_kC_l + h_jh_lC_k + h_kh_lC_j + 3E_jh_{kl} + 3E_kh_{jl} + 3E_lh_{jk}\Big\}, \tag{3.1}$$

where

$$\rho := \phi(\phi - s\phi), \quad h_j := \alpha b_j - sy_j, \quad h_{jk} := \alpha^2 a_{jk} - y_jy_k, \quad C_j := (X_4r_{00} + Y_4\alpha s_0)h_j + 3\Lambda J_j,$$
$$\mu := \frac{-1}{3}(Q - sQ'), \quad J_j := \alpha^2(s_{j0} + \frac{r_{j0}}{\Delta} + \frac{-Q}{\Delta}\alpha s_j) - (\frac{r_{00}}{\Delta} + \Pi\alpha s_0)y_j,$$
$$\Lambda := -Q'', \quad E_j := (X_6r_{00} + Y_6\alpha s_0)h_j + 3\mu J_j),$$
$$Y4 := -2QX_4 + \frac{3Q'Q''}{\Delta}, \quad Y6 := -2QX_6 + \frac{Q'(Q - sQ')}{\Delta},$$
$$X4 := \frac{1}{(2\Delta^2)}(-2\Delta Q''' + 3(Q - sQ')Q'' + 3(B - s^2)(Q'')^2),$$
$$X6 := \frac{1}{2\Delta^2}((Q - sQ')^2 + \Big(2(s + BQ) - (B - s^2)(Q - sQ')\Big)Q'').$$

Let $F=\alpha\phi(s)$, $s=\beta/\alpha$, be an (α,β)-metric on an n-dimensional manifold M. Then the S-curvature of F is given by

$$\mathbf{S}=\Big[2\Psi-\frac{f'(b)}{bf(b)}\Big](s_0+r_0)-\frac{\Phi}{2\Delta^2\alpha}(r_{00}-2Q\alpha s_0), \tag{3.2}$$

where

$$f(b):=\frac{\int_0^\pi \sin^{n-2}t\ T(b\cos t)dt}{\int_0^\pi \sin^{n-2}tdt},\quad T(s):=\phi(\phi-s\phi')^{n-2}\big[(\phi-s\phi')+(b^2-s^2)\phi''\big].$$

Here, we calculate the S-curvature of 5-th root (α,β)-metric and obtain the following.

Lemma 3.2. *The S-curvature of* 5*-th root* (α,β)*-metric is given by*

$$\begin{aligned}
\mathbf{S}=&\frac{1}{2s^2AB}\Big\{3c_2^2s^2+13c_2c_3s^4+10c_3^2s^6+3c_1c_2+10c_1c_3s^2-\frac{f'(b)}{bf(b)}\Big\}\big(s_0+r_0\big)\\
&-\frac{1}{4\alpha A^2B^2s^3}\Big\{8c_2c_3^3s^{12}-3c_2^4b^2s^4-20c_3^4b^2s^{12}-60nc_1c_2^2c_3s^6-112nc_2c_1c_3^2s^8\\
&+36nc_1c_2^4b^2s^6+120nc_1c_2c_3^3b^2s^{12}+640nc_1c_3^4b^2s^{14}+38nc_2^4c_3b^2s^{10}+12nc_2^3c_3^2b^2s^{12}\\
&+2456nb^2c_2^2c_3^3s^{14}+2240nb^2c_2c_3^4s^{16}-56nc_1^2c_2^2c_3s^8-12nc_1^2c_2c_3^2s^{10}-30nc_1c_2^3c_3s^{10}\\
&-94nc_1c_2^2c_3^2s^{12}-120nc_1c_2c_3^3s^{14}-6c_2^4s^6-20nc_1^2c_2c_3s^4-58c_1c_3b^2c_2^2s^4-136c_1c_3^2s^6\\
&-42nc_2^3c_3s^8-10nc_2^2c_3^2s^{10}-10nc_2c_3^3s^{12}-64nc_3^3c_1s^{10}-34b^2c_1^2c_2c_3s^2+4c_1^2c_2c_3s^4\\
&+80nc_3^5b^2s^{18}-4nc_1^2c_3^2s^6+36nb^2c_2^5s^8+4c_1c_2^3s^4-80nc_3^5s^{20}+4c_1^2c_2^2s^2-8nc_1^2c_2^3s^6\\
&-10nc_2^3c_1s^4-24nc_2^2c_3^3s^{16}-36nc_1c_2^4s^8-96nc_1^2c_3^3s^{12}-64nc_1c_3^4s^{16}-38nc_2^4c_3s^{12}\\
&-62c_2c_3^3b^2s^{10}-2240nc_2c_3^4s^{18}+28c_1c_2^2c_3s^6+80c_1c_2c_3^2s^8+8c_2^2c_3^2s^{10}-9c_1c_2^3b^2s^2\\
&-21c_2^3c_3b^2s^6-60b^2c_2^2c_3^2s^8-80c_1c_3^3b^2s^8-60c_1^2c_3^2b^2s^4-40c_3^4s^{14}-130nc_2^3c_3^2s^{14}\\
&-36c_2^5s^{10}+48c_1c_3^3s^{10}+48c_1^2c_3^2s^6+56b^2c_1^2c_2^2c_3s^6+128nc_1^2c_2c_3^2b^2s^8-4nc_1^2c_2^2s^2\\
&+8nb^2c_1^2c_2^3s^4+944nb^2c_1c_2^2c_3^2s^{10}+304nb^2c_1c_2^3c_3s^8\\
&+96nc_1^2c_3^3b^2s^{10}\Big\}\Big(r_{00}+(c_1+3c_2s^2+5c_3s^4)s_0\Big),
\end{aligned}$$

(3.3)

where

$$\begin{aligned}
A:=&-c_2s^2-c_3s^4+2c_1c_2b^2s^2+4c_1c_3b^2s^4+6c_2^2b^2s^4+22c_2c_3b^2s^6+20c_3^2b^2s^8\\
&-2c_1c_2s^4-4c_1c_3s^6-6c_2^2s^6-22c_2c_3s^8-20c_3^2s^{10},\\
B:=&c_2+2c_3s^2,\\
T:=&(c_1s+c_2s^3+c_3s^5)\big(c_1s+c_2s^3+c_3s^5-s(c_1+3c_2s^2+5c_3s^4)\big)^{n-2}\Big\{c_1s\\
&+c_2s^3+c_3s^5-s(c_1+3c_2s^2+5c_3s^4)+(b^2-s^2)(6c_2s+20c_3s^3)\Big\}.
\end{aligned}$$

Now, we study weakly Landsberg 5-th root (α,β)-metrics and prove the following.

Theorem 3.3. *Let* $F=\sqrt[5]{c_1\alpha^4\beta+c_2\alpha^2\beta^3+c_3\beta^5}$ *be a Landsberg* (α,β)*-metric. Then* F *has vanishing* S*-curvature.*

Proof. Suppose F is Landsberg, that is

$$\mathbf{L} = 0. \tag{3.4}$$

By Lemmas 3.1, we calculate L_{jkl} and contract L_{jkl} with $b^j b^k b^l$, we have

$$\mathbf{L} = f_2\alpha^4 + f_1\alpha^2 + f_0 = 0, \tag{3.5}$$

where

$$\begin{aligned} f_2 :&= 407r_{00}c_2^2 - 368r_{00}c_3Bc_2 + 656\beta s_0c_2c_3 - 120B\beta c_3^2r_0 - 250\beta s_0c_3^2b^2 + 30B^2c_3^2r_{00} \\ &\quad + 200Br_{00}b^2c_3^2 + 48\beta c_2c_3r_0 - 440c_2c_3r_{00}b^2 + 432r_{00}c_3c_1 - 115\beta s_0c_3^2B, \\ f_1 :&= -70\beta^2r_{00}c_3^2B - 100\beta^2c_3^2r_{00}b^2 + 266\beta^2r_{00}c_2c_3 + 150\beta^3s_0c_3^2, \\ f_0 :&= 60\beta^4r_{00}c_3^2. \end{aligned}$$

By assumption, we have $L = 0$. Using (3.5) imply that there exists a non-zero function $g = g(x, y)$ of degree 4 in y such that

$$\beta^4r_{00}c_3^2 = g\alpha^2. \tag{3.6}$$

This contradicts with the positive-definiteness of α. Thus $r_{ij} = 0$. Putting it into (3.5), we obtain

$$\mathbf{L} = s_0(-656c_2 + 250b^2c_3 + 115c_3B)\alpha^2 - 150s_0\beta^2c_3 = 0. \tag{3.7}$$

Similarly, (3.7) implies that there exists a non-zero function $h = h(x, y)$ of degree 1 in y such that the following holds

$$s_0\beta^2c_3 = h\alpha^2. \tag{3.8}$$

This contradicts with the positive-definiteness of α. Thus $s_i = 0$. By substituting $r_{ij} = 0$ and $s_i = 0$ for (3.3), we find

$$\mathbf{S} = 0. \tag{3.9}$$

the proof is complete. □

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ON A CLASS OF QUASI-EINSTIEN(α, β)-METRICS

SAEEDEH MASOUMI AND BAHMAN REZAEI

ABSTRACT. In this paper, we introduce the notion of quasi-Ricci and weakly quasi-Einestien for Finsler metrics, which is the combination of the Ricci curvature and the S-curvature. We study quasi-Einestien kropina metric on a manifold of dimensional $n \geq 2$. Furthermore, by supposing a quasi-Einestien Kropina metric, we find sufficient conditions under which a Kropina metric has quasi-Ricci flat.

Key words and phrases: Finsler metric; (α, β)- metric; quasi-Einetien; quasi-Ricci flat.

1. Introduction

In 2012, Zhang and Shen introduced the condition of Einestien Kropina metric [11]. They proved a non-Riemannian Kropina metric $F = \frac{\alpha^2}{\beta}$ with constant Killing from β on a manifold M with dimensional $n \geq 2$, is an Einstien metric if and only if Riemannian metric α is an Einestien metric. J. Case and Y. Shu , G. Wei studeid m-quasi-Einestien on manifold Reimannein [15, 14, 7].
Recently, Ohta introduced a definition of N-Ricci curvature [5]. Quasi-Einstein Finsler metric is a generalization of Einstein metric in Finsler geometry, which is investigated by H.Zhu [13]. We will generalize N-quasi Einestein in Finsler geometry [2, 5]. Suppose (M, F) is manifold Finsler n- dimensional with measure $dV_F = e^{-f} dV_{BH}$, is called N- weakly quasi-Einstein if it satisfies

$$Ric + \dot{S} - \frac{S^2}{N-n} = (n-1)(c + \frac{3\theta}{F})F^2,$$

where $\dot{S}$ is the covariant derivative of S along a geodesic of F and $c = c(x)$ is scalar function and θ is a 1- form on M. If $\theta = 0$ and $N = \infty$,then Finsler metric F is called quasi-Einstein. Furthermore, quasi-Einstein Finsler metric F be is called quasi-Ricci flat if $c = 0$.
In 2022, Zhu studied quasi-Einstein square metrics. He found the structure of quasi-Ricci flat square metric which is the famous Berwalds metric.
In 2007, Li-Shen studied (α, β)-metrics of constant flag curvature [4, 1]. The Ricci-curvature and S-curvature have important and fundamental topic in Finsler Geometry [9, 8, 10].
In this paper, we are going to study quasi-Einestien Kropina metrics. Firstly, we verify essential conditions for a Kropina metric. Finally, we determine the structure of quasi-Einstein and quasi-Ricci flat for the Kropina metric. The main theorem is as follows

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Theorem 1.1. *Let $F = \frac{\alpha^2}{\beta}$ be a Kropina metric on n-dimensional manifold M with volum form $dV = e^{-f} dV_\alpha$. Then it is a quasi-Einstein Finsler metric if and only if*

$$s^i_j s^j_i = -2\big[2c(n-1) + s^i s_i\big], \tag{1.1}$$

CaseI:*Assume $n \neq 2$*

A: *if Finsler metric F be regullar then:*

$$Ric_\alpha = \frac{1}{B^2}\Big[(n-2)\big(s_0^2 - \sigma^2\beta^2\big) - 2(n-2)\sigma s_0 \beta\Big] - \frac{1}{B}\Big[2s_{0|0} - 2f_0 s_0\Big] - f_{0|0} + \eta\alpha^2,$$

B: *if Finsler metric F be singullar then*

$$Ric_\alpha = (n-2)\bigg(s_0^2 - \sigma^2\beta^2\bigg) - 2(n-2)\sigma s_0\beta - 2s_{0|0} + 2f_0 s_0 - f_{0|0} + \eta\alpha^2.$$

2. Preliminaries

Let (M, F) be a Finsler space.A spray on M is a smooth vector field G on tangent space TM_0 expressed in a standard local coordinate system (x^i, y^i) in TM as follows $G(x,y) = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial y^i}$, where $G^i(x,y)$ are local functions on TM satisfying

$$G^i(x, \lambda y) = \lambda^2 G^i(x,y), \quad \lambda > 0.$$

The Riemann curvature $R_y = R^i_k(y)\frac{\partial G^i}{\partial x^i} \otimes dx^k$ is expressed

$$R^i_k := 2\frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

Ricci curvature is the trace of the Riemann curvature, which is called by $Ric := R^m_m$. (α, β)-metric is a special class of Finsler metrics and can be defined by the form $F := \alpha\phi(s), \quad s = \frac{\alpha}{\beta}$, where α is Riemann metric and β is one form. It is known that is positive and strongly convex on TM_0 if and only if

$$\phi(s) - s\phi'(s) + \big(B - s^2\big)\phi''(s) > 0,$$

where $B := a^{ij}b_i b_j = \|\beta\|^2_\alpha$. The spray coefficients of (α, β)-metrics are given by [3]

$$G^i = G^i_\alpha + Q^i,$$

where $Q^i := \alpha Q s^i_0 + \theta\bigg(r_{00} - 2\alpha Q s_0\bigg)\frac{y^i}{\alpha} + \psi\bigg(r_{00} - 2\alpha Q s_0\bigg)b^i$, and

$$Q = \frac{\phi'}{\phi - s\phi'}, \quad \theta = \frac{\big(\phi - s\phi'\big)\phi' - s\phi'\phi''}{2\phi\Big[\phi - s\phi' + (B - s^2)\phi''\Big]},$$

$$\psi = \frac{\phi''}{2\Big[\phi - s\phi' + (B - s^2)\phi''\Big]}, \quad G^i_\alpha = \frac{1}{4}a^{ij}\Big[[\alpha^2]_{x^l y^j} y^k - [\alpha^2]_{x^j}\Big],$$

are the spray coefficients of the Riemannian metric α.

Definition 2.1. *Let F be measureable Finsler metric with volum form $dV_F = e^{-f} dV_{BH}$ on TM_0. Then combination of the Ricci curvature and S-curvature is called **quasi-Rcci curvature** by*

$$\boldsymbol{Qric} := Ric + \dot{S}, \tag{2.1}$$

where $\dot{S}$ *is the covariant derivative of* S *along a geodesic of* F *and* $Ric := R^m_m$.

3. Quasi-Einestien-Kropina metric

Lemma 3.1. *Let* $F = \frac{\alpha^2}{\beta}$ *be a kropina metric an* $n-$*dimesional manifold* M. *Then quasi-Ricci curvature of* F *is given by*

$$\begin{aligned} Ric+\dot{S}=&Ric_\alpha + f_{0|0} - \frac{s^i_j s^j_i F^2}{4} + F\big(f_{x^i}s^i_0 - s^i_{0|i}\big) - \frac{1}{B^2}\big((n-2)s_0^2 + 2r_{00}r \\ &+ r_0^2 - 8r_0 s_0 + 2rs_0F\big) + \frac{1}{B}\Bigg[r^i_i r_{00} + r_{00|b} + r_{i0|0}b^i - r_{0i}r^i_0 - 3r_{0i}s^i_0 - 2s_{0|0} \\ &- \frac{3}{2}\alpha s_i s^i_0 F - r_i s^i_0 F - \frac{1}{2}s_{i|0}Fb^i + +s_{0|b}F + s_0 r^i_i F - \frac{3}{2}r^i_0 F \\ &- \frac{s_i s^i F^2}{2} + r_i s^i_0 F + r_{0|0} - f_b r_{00} - f_b s_0 F + 2f_0 s_0\Bigg] - \frac{(n+7)}{F^2B^2}r^2_{00} \\ &- \frac{1}{F}\Bigg[2(n+3)r_{00}s_0 - \frac{10r_{00}r_0}{B^2} + \frac{2r_{00|0} - 2f_0 r_{00}}{B}\Bigg]. \end{aligned} \tag{3.1}$$

Proof. The spray coefficients of $F = \frac{\alpha^2}{\beta}$ are given by

$$G^i = G^i_\alpha + \alpha Q s^i_0 + \theta(r_{00} - 2\alpha Q s_0)\frac{y^i}{\alpha} + \Psi(r_{00} - 2\alpha Q s_0)b^i, \tag{3.2}$$

where $Q = -\frac{1}{2s}, \quad \psi = \frac{1}{2B}, \quad \theta = -\frac{s}{B}$.

By (2.1) and consider the Lemma [13] and proposition [12], we can show (3.1). □

Now we have the following proof of theorem 1.1.

Proof. Let F be a quasi-Einestein kropina metric. Then, by means quasi-Einestein and lemma 3.1. we have

$$\overset{\alpha}{Ric} + T^i_i + \dot{S} - (n-1)cF^2 = 0,$$

where $c = c(x)$ is a scalar function. we have

CaseI: Assume $n \neq 2$. In this case, we show

$$Ric_\alpha = \frac{1}{B^2}\big[(n-2)\big(s_0^2 - \sigma^2\beta^2\big) - 2(n-2)\sigma s_0\beta\big] - \frac{1}{B}\big[2s_{0|0} - 2f_0 s_0\big] - f_{0|0} + \eta\alpha^2.$$

where $\eta = \eta(x)$is a scalar function. □

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HARMONIC VECTOR FIELDS IN FINSLER GEOMETRY

MIR AHMAD MIRSHAFEAZADEH AND BEHROZ BIDABAD

Abstract. The article discusses various mathematical concepts related to Finsler manifolds, which are spaces that have a metric that varies depending on direction. The authors define the horizontal differential, divergence, and p-harmonic form on such manifolds. They then prove a theorem that relates harmonic forms to the vanishing of the horizontal Laplacian. This leads to a new way of defining harmonic vector fields in Finsler geometry. The article also presents a Bochner-Yano type classification theorem based on the harmonic Ricci scalar. Finally, the authors demonstrate that closed orientable Finsler manifolds with a positive harmonic Ricci scalar have zero Betti number, which is a topological invariant that measures the number of holes in a space.

Key words and phrases: Finsler geometry; Harmonic vector field; Betti number.

1. Introduction

The Hodge Laplacian operator is a differential operator that acts on differential forms on a manifold. It is defined as the sum of the exterior derivative and its adjoint, followed by the codifferential and its adjoint. In other words, it is the composition of the Laplace-Beltrami operator and the Hodge star operator.

One of the main applications of the Hodge Laplacian operator is in the study of harmonic forms and harmonic maps. A differential form ω is called harmonic if it satisfies the equation $\Delta\omega = 0$, where Δ is the Hodge Laplacian operator. Harmonic forms play an important role in various areas of mathematics and physics, including topology, geometry, and quantum field theory.

Another application of the Hodge Laplacian operator is in the study of eigenvalues and eigenfunctions of Laplace-Beltrami operators. The eigenvalues of the Hodge Laplacian operator are related to the spectrum of the Laplace-Beltrami operator, and they provide important information about the geometry and topology of the underlying manifold.

Overall, the Hodge Laplacian operator is a powerful tool in differential geometry and its applications, providing insights into the structure and behavior of geometric objects on manifolds [4].

In order to extend the notion of a harmonic vector field to Finsler manifolds, we need to first define what we mean by the Laplace-Beltrami operator on such manifolds.

For a Riemannian manifold, the Laplace-Beltrami operator is defined as the divergence of the gradient of a function. However, in Finsler geometry, there is no

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natural notion of a gradient, since the metric structure is not necessarily induced by a smooth inner product.

One way to overcome this difficulty is to use the concept of a spray. A spray is a vector field on a Finsler manifold that satisfies certain properties, such as being tangent to the unit sphere at each point and having a unique geodesic passing through each point with a given initial velocity.

Using sprays, we can define a Finsler Laplacian operator as follows: given a smooth vector field X on a Finsler manifold M, we can define its Finsler divergence $div(X)$ as the trace of the covariant derivative of X with respect to the spray. Then, we can define the Finsler Laplacian of a smooth function f on M as $\Delta f = div(grad(f))$, where $grad(f)$ is the gradient of f with respect to the spray.

With this definition, we can extend the notion of a harmonic vector field to Finsler manifolds by requiring that its divergence is zero with respect to the Finsler Laplacian operator. That is, a vector field X on a Finsler manifold M is said to be harmonic if $div(X) = \Delta f(X) = 0$, for some smooth function f on M. This definition captures the idea that a harmonic vector field on a Finsler manifold preserves the harmonic structure of the manifold in a way that is compatible with the Finsler metric [3].

The Betti number is a numerical invariant that measures the number of holes in a topological space. It is named after Enrico Betti, an Italian mathematician who introduced it in the 19th century. The Betti number is defined as the rank of the corresponding homology group, which is a mathematical tool used to study the structure of spaces.

The Betti number has many applications in various fields of mathematics and science. For example, in algebraic geometry, it is used to study the topology of algebraic varieties, which are geometric objects defined by polynomial equations. In topology, the Betti number is used to distinguish different types of surfaces and higher-dimensional spaces. In physics, it has applications in the study of phase transitions and the behavior of materials.

One important application of the Betti number is in the classification of manifolds. A manifold is a mathematical object that locally looks like Euclidean space, and it can have a nontrivial topology. The Betti numbers of a manifold provide information about its topology and can be used to distinguish between different types of manifolds. For example, the Betti numbers of a sphere are different from those of a torus, which in turn are different from those of a Klein bottle.

Overall, the Betti number is a powerful tool in topology and geometry that helps us understand the structure of spaces and their properties. Harmonic vector fields have applications in various fields of physics and mathematics, including fluid mechanics, electromagnetism, and differential geometry. In fluid mechanics, harmonic vector fields correspond to the steady-state flow of a fluid, and are used to model the behavior of fluids in various settings. In electromagnetism, they are used to describe the behavior of electromagnetic fields in a vacuum. In differential geometry, harmonic vector fields are used to study the geometry of Riemannian manifolds, and have applications in the study of minimal surfaces and the theory of relativity. Overall, harmonic vector fields are a powerful tool for understanding the behavior of physical systems and the geometry of mathematical spaces [2].

2. Main results

Definition 2.1. *Let (M, F) be a Finsler manifold and*

$$\varphi = \frac{1}{p!}\varphi_{i_1\cdots i_p}(z)dx^{i_1} \wedge \cdots \wedge dx^{i_p} \in \Lambda_p^H,$$

a horizontal p-form on SM. A horizontal differential operator *is a differential operator on SM given by*

$$\begin{cases} d_H : & \Lambda_p^H \longrightarrow \Lambda_{p+1}^H \\ & \varphi \longmapsto d_H\,\varphi \end{cases} \tag{2.1}$$

where, for $1 \le i, i_k \le n$ and $1 \le k \le p$, we have

$$d_H\,\varphi = \frac{1}{(p+1)!}\big(\nabla_i\varphi_{i_1\cdots i_p} - \nabla_{i_1}\varphi_{ii_2\cdots i_p} - \cdots - \nabla_{i_p}\varphi_{i_1\cdots i_{p-1}i}\big)dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}. \tag{2.2}$$

Let φ and π be the two arbitraries horizontal p-forms on SM with the components $\varphi_{i_1\cdots i_p}$ and $\pi_{i_1\cdots i_p}$, respectively. We consider an inner product $(.,.)$ on $\Lambda_p^{\mathtt{H}}$ as follows

$$(\varphi, \pi) := \int_{SM} \frac{1}{p!}\ \varphi^{i_1\cdots i_p}\ \pi_{i_1\cdots i_p}\ \eta, \tag{2.3}$$

where, $\varphi^{i_1\cdots i_p} = g^{i_1j_1}\cdots g^{i_pj_p}\varphi_{j_1\cdots j_p}$.

Definition 2.2. *Let (M, F) be a Finsler manifold and ψ a horizontal $(p+1)$-form on SM, given by*

$$\psi = \frac{1}{(p+1)!}\psi_{ii_1\cdots i_p}dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}.$$

We define the horizontal divergence *(co-differential) of ψ by*

$$(\delta_H\,\psi)_{j_1\cdots j_p} := -g^{ij}(\nabla_i\psi_{jj_1\cdots j_p} - \psi_{jj_1\cdots j_p}\nabla_0 T_i). \tag{2.4}$$

Definition 2.3. *Let (M, F) be a Finsler manifold. A* horizontal Laplacian *on SM is defined by*

$$\Delta_H := d_H\delta_H + \delta_H d_H, \tag{2.5}$$

where d_H and δ_H are horizontal differential and horizontal co-differential operators on SM, respectively.

Theorem 2.4 ([1])**.** *Let (M, F) be a closed Finsler manifold. If ω is a horizontal p-form on SM, then*

$$\Delta_H\,\omega = 0 \quad \text{if and only if} \quad d_H\,\omega = 0, \text{ and } \quad \delta_H\,\omega = 0. \tag{2.6}$$

Definition 2.5. *A horizontal p-form φ on SM is called* horizontally harmonic *if we have*

$$\Delta_H\,\varphi = 0.$$

The horizontal harmonic p-forms will be referred to in the suite as h-harmonic p-forms or simply h-harmonic.

Definition 2.6. *Let (M, F) be a Finsler manifold. A vector field $X = X^i(x)\frac{\partial}{\partial x^i}$ on M is called* harmonic *related to the Finsler structure F if the associate horizontal 1-form $X = X_i(z)dx^i$ is h-harmonic on SM.*

Theorem 2.7 ([1])**.** *Let (M, F) be a Finsler manifold. Every cohomology class $H^1(M)$ contains a unique harmonic representative.*

Let $X = X^i(x)\frac{\partial}{\partial x^i}$ be a vector field on (M, F). We define the *harmonic Ricci scalar* $\tilde{Ric}$ as follows

$$\tilde{Ric}(X, X) := X^k X^t R_{tk} - X^k \dot{\nabla}_r X^j R^r_{jk} - X^k \nabla_k X^j \nabla_0 T_j. \tag{2.7}$$

Theorem 2.8 ([1])**.** *Let (M, F) be a closed Finsler manifold and X a harmonic vector field on M.*

1. *If $\tilde{Ric} = 0$, then X is parallel.*
2. *If $\tilde{Ric} > 0$, then X vanishes.*

Theorem 2.9 ([1])**.** *In a closed orientable Finsler manifold with a positive harmonic Ricci scalar $\tilde{Ric} > 0$, the first Betti number vanishes.*

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ON QUASI-EINSTEIN RANDERS METRIC

ILLATRA KHAMONEZHAD, BAHMAN REZAEI, AND MEHRAN GABRANI

ABSTRACT. In this pepar we study the special form of wieghed Ricci Curvature which called Quasi-Einstein Finsler metric in Randers metrics.
Key words and phrases: Quasi-Einstein Finsler metric; Randers metric; S-curvature

1. Introduction

In Finsler manifold we would like to have a measure. Riemannian manifold have a unique canonical measure, but chossing measure in Finsler manifold isnt simple. Let $(M, F, \mathfrak{m})$ be a Finsler measured manifold, where (M, F) be a Finsler manifold with metric F and $\mathfrak{m}$ be a positive C^{∞} on M. For $N \in \mathbb{R}/\{n\}$ Ohta intorduced Finsler Weighed Ricci Curvature as following

$$Ric_N(x) := Ric(x) + \psi''_{\eta}(0) - \frac{\psi'_{\eta}(0)^2}{N-n},$$

where ψ and η respectively are C^{∞} in $\mathbb{R}$ and M [2].
Let $N \to \infty$, then the following arise;

$$Ric_{\infty}(x) = Ric(x) + \psi''_{\eta}(0).$$

Z. Shen in 1997 introduced new quantity that called S-curvature[6].
Substitute $\psi'_{\eta}(0)$ with quantity $S(x)$ apear the following equation,

$$Ric_{\infty}(x) = Ric(x) + \dot{S}(x). \tag{1.1}$$

That is first studed by otha [3]. The projective Ricci curvature is Weighed Ricci curvature that is projecivly invariant when the volume form is fixed [4]. Another weighted Ricci curvature is (a, b)-weighted Ricci curvature in Finsler geometry that define by Z.Shen and R.Zhao [5]. In special case a Finsler metric F called called $Quasi - Einstein$ Finsler metric if F satisfies $Ric_{\infty} = (n-1)cF^2$ [7].

Theorem 1.1. *Let $F = \alpha + \beta$ be the Randers metric on manifold M of dimension $n \geq 3$ with volume form dV be a volume form. F is a quasi-Einstein Finsler metrics if and only if F satisfy following*

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\begin{aligned}
(1.2)\qquad e_{00} =&\sigma(x)(\alpha^2-\beta^2),\\
(1.3)\qquad \widetilde{Ric}_{00} =&(\alpha^2+3\beta^2)(n+1)c(x)-2\beta s^i_{0|i}-2s^i_0 s_{i0}+4s^i_0 r_{i0}-2f_0 s_0+\alpha^2 s^i_j s^j_i\\
&+2s_{0|0}-f_{0|0}+(n+1)s_0^2+2\beta f_{x^i}s^i_0+\sigma^2(\alpha^2+\beta^2)\frac{(n+7)}{4},\\
(1.4)\qquad s^i_{0|i} =&\beta(n+1)c(x)-2s_i s^i_0+f_{x^i}s^i_0-\frac{1}{2}\sigma_{|0}\\
&+\frac{1}{2}f_0\sigma-\frac{(n+7)}{2}\sigma s_0+\sigma(\sigma\beta+s_0)+\frac{(n+7)}{4}\sigma^2\beta.
\end{aligned}
$$

2. Preliminaries

Let (M,F) be a Finsler metric. The non-negative function F on TM is a Finsler metric of M (or Finsler structure) if satisfying three conditions:
(i) Regularity, (ii) Positive 1-homogeneity and (iii) Strong convexity.
Define the most fundamental measure in Finsler geomatry named $Busemann-Hausdorff$ on M by

$$
\mathfrak{m}_{BH}(dx) := \Phi_{BH}(x)dx^1dx^2\dots dx^n, \tag{2.1}
$$

where the function Φ_{BH} is defined as following

$$
\frac{\omega_n}{\Phi_{BH}(x)} = \mathfrak{L}\Bigg(\Bigg\{(a_i)_{i=1}\in\mathbb{R}\Bigg|F\Bigg(\sum_{i=1}^{n}a_i\frac{\partial}{\partial x^i}\Bigg|_x\Bigg)<1\Bigg\}\Bigg). \tag{2.2}
$$

In local cordinates, a volume form is $dV=\sigma(x)dx^1\dots dx^n$, where σ is positive function. the quantity S is measure by distortions rate of change along geodesics where destortions $\tau(x,y)$ is defined as following

$$
\tau(x,y) := ln\frac{\sqrt{detg_{ij}(x,y)}}{\sigma(x)}.
$$

The S- curvature and $\dot{S}$ are defined by

$$
\begin{aligned}
S(x,y) &:= \frac{d}{dt}[\tau(c(t),\dot{c}(t))]\Big|_{t=0} \quad or \quad S(x,y):=\tau_{|i}(x,y)y^i,\\
\dot{S}(x,y) &:= \frac{d}{dt}[S(c(t),\dot{c}(t))]\Big|_{t=0} \quad or \quad \dot{S}(x,y):=S_{|i}(x,y)y^i,
\end{aligned}
$$

Where $c=c(x)$ be the geodesic with $c(0)=x$ and $\dot{c}=y$ and "—" denotes the horizontal covariant derivative with respect to F. A vector field G, induced by a Finsler metric F on TM_0, is given by

$$
G = y^i\frac{\partial}{\partial x^i}-2G^i(x,y)\frac{\partial}{\partial x^i},
$$

which is called spray of F and $G^i(x,y)$, are local functions on TM_0, satisfying $G^i(x,\lambda y)=\lambda^2G^i(x,y)$, with $\lambda>0$ is called spray coefficients of F. In general the spray cofficients

The famous and importand family of Finsler mertic are $(\alpha,\beta)-metrics$ that can be express by the form

$$
F=\alpha\phi(s), \quad s=\frac{\beta}{\alpha}.
$$

Put $\phi(s) = 1 + s$, then $F = \alpha + \beta$ which is called Randers metric.
The spray cofficients of $(\alpha, \beta) - metrics$ are given by

$$G^i = G^i_\alpha + Q^i,$$

where

$$G^i_\alpha = \frac{1}{4} a^{ij} \{\alpha_{x^k y^j} y^j y^k - \alpha_{x^j}\},$$

$$Q^i = \alpha Q s^i_0 + \Theta(r_{00} - 2\alpha Q s_0) \frac{y^i}{\alpha} + \Psi(r_{00} - 2\alpha Q s_0) b^i, \tag{2.3}$$

where

$$Q = \frac{\phi'}{\phi - s\phi'}, \quad \Theta = \frac{(\phi - s\phi')\phi' - s\phi'\phi''}{2\phi\left[\phi - s\phi' + (B - s^2)\phi''\right]}, \quad \Psi = \frac{\phi''}{2\left[\phi - s\phi' + (B - s^2)\phi''\right]}.$$

let (M, F) be a n-dimensional Finsler manifold and the geodesic coefficients of F denote by G^i. Define $R = R^k_i(x, y) dx^i \otimes \frac{\partial}{\partial x^i}|x$ by

$$R^i_k = 2\frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

The family $R := R^k_i$ is called the Riemann curvature (or flag curvature tensor). The *Ricci* scalar is defined by $Ric := R^i_i$ and a Finsler metric F on M is called an Einstein metric if there is function λ defined on M such that $Ric = \lambda(x)F^2$.

3. Proof of Theorem

In this section we discuss the necessery and sufficient condition that Randers metric be a Quasi-Eistain Finsler metric. Before prove of Theorem 1.1 we need the following lemma;

Lemma 3.1. *Randers metric F is Quasi-Einstein if and only if Rat and Irrat are equal with zero.*

Proof. Assume Randers metric is Qasi-Einstein metric, $Ric + \dot{S} = (n-1)cF$, so we have the following

$$\begin{aligned}
0 &= {}^\alpha Ric + I^i_i + \dot{S} - (n-1)cF \\
&= {}^\alpha Ric + \frac{(n+7)}{F^2}\left\{\frac{-1}{4} r_{00} + \alpha r_{00} s_0 - \alpha^2 s_0^2\right\} \\
&+ \frac{1}{F}\left\{2\alpha f_0 s_0 - 4\alpha r_{0i} s^i_0 + r_{00|0} - 2\alpha s_{0|0} + 4\alpha^2 s_i s^i_0 - f_0 r_{00}\right\} \\
&+ \left\{2\alpha s^i_{0|i} - 2s_{0i} s^i_0 - \alpha^2 s^i_j s^j_i - 2\alpha f_{x^i} s^i_0 + f_{0|0} - (n-1)cF^2\right\}
\end{aligned} \tag{3.1}$$

By multiplaying (3.1) by F^2 removes y from the denominators. So Rat and Irrat are the following form,

$$\begin{aligned} Rat =&(\alpha+\beta)\left\{{}^{\alpha}Ric - 2s_{i0}s^i_0 - \alpha^2 s^i_j s^j_i + f_{0|0}\right\} \\ &+\alpha^2\left\{4\beta s_i s^i_0 - 4r_{0i}s^i_0 + 2f_0 s_0 + 4\beta s^i_{0|i} - (n+7)s^2_0\right\} \\ &+\beta\left\{r_{00|0} - f_0 r_{00}\right\} - \frac{(n+7)}{2}r_{00} \\ &-(\alpha^4+6\alpha^2\beta^2+\beta^4)(n-1)c(x), \end{aligned} \tag{3.2}$$

$$\begin{aligned} Irrat =&2\beta^{\alpha}Ric + 2(\alpha^2+\beta^2)s^i_{0|i} - 4(n-1)c(x)\beta(\alpha^2+\beta^2) \\ &+4\beta s^i_0 s_{i0} - 2\beta s_{0|0} - 4\beta r_{0i}s^i_0 - 2(\alpha^2+\beta^2) + r_{00|0} \\ &-2\beta f_{0|0} - f_0 r_{00} + 2\beta f_0 s_0 + (n+7)r_{00}s_0 + 4\alpha^2 s_i s^i_0 \\ &-2(\beta^2+\alpha^2)f_{x^i}s^i_0 - 2\beta\alpha^2 s^i_j d^j_i. \end{aligned} \tag{3.3}$$

The condition for a Randers metric to be Quasi-Einstein metric is to satisfy the following eqation,

$$Rat + \alpha Irrat = 0, \tag{3.4}$$

the rest proof that Rat and Irrat must be zero is similar to [1] so we omit it. □

Proof of Theorem 1.1 Now these two Equation we discuss another necessery condition Let F is Quasi-Einstein metric. So,

$$\begin{aligned} 0(3.5)&= Rat - \beta Irrat \\ &= (\alpha^2-\beta^2)\{{}^{\alpha}Ric - (\alpha^2+3\beta)(n-1)c(x) + 2f_0 s_0 + 2\beta s^i_{0|i} + 2s^i_0 s_{i0} \\ &-4s^i_0 r_{i0} - \alpha^2 s^i_j s^j_i - 2s_{0|0} + f_{0|0} - (n+7)s^2_0 - 2\beta f_{x^i}s^i_0\} - \frac{(n+7)}{4}(r_{00}+2\beta s)^2. \end{aligned}$$

Thus we have

$$r_{00} + 2\beta s_0 = \sigma(x)(\alpha^2-\beta^2), \tag{3.6}$$

where $\sigma(x)$ is a scalar function on M.
Return and put (19) to $Rat - \beta Irrat = 0$, by divide off factor $(\alpha^2-\beta^2)$ imply following result .

$$\begin{aligned} {}^{\alpha}Ric_{00} =&(\alpha^2+3\beta)(n-1)c(x) - 2f_0 s_0 - 2\beta s^i_{0|i} - 2s^i_0 s_{i0} \\ &+4s^i_0 r_{i0} + \alpha^2 s^i_j s^j_i + 2s_{0|0} - f_{0|0} + (n+7)s^2_0 + 2\beta f_{x^i}s^i_0 \\ &+\frac{(n+7)}{4}(\alpha^2-\beta^2)\sigma^2(x). \end{aligned} \tag{3.7}$$

Back to $Irrat = 0$. Replace "${}^{\alpha}Ric_{00}$", "r_{00}", "$r_{00|0}$" and "$s^i_0 r_{i0}$". Thus we have

$$\begin{aligned} s^i_{0|i} =&\beta(n+1)c(x) - 2s_i s^i_0 + f_{x^i}s^i_0 - \frac{1}{2}\sigma_{|0} \\ &+\frac{1}{2}f_0\sigma - \frac{(n+7)}{2}\sigma s_0 + \sigma(\sigma\beta + s_0) + \frac{(n+7)}{4}\sigma^2\beta. \end{aligned} \tag{3.8}$$

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HOMOTOPICALLY COVERING HOMOTOPY PROPERTY

ALI PAKDAMAN AND SABA DEHROOYE

Abstract. In this paper we generalize covering homotopy property by homotopically covering homotopy property and study its effect on other concepts such as fibrations and unique path lifting.

Key words and phrases: Covering Homotopy Property; Homotopically Covering Homotopy Property; Fibration.

1. Introduction

A map $p : E \to B$ has the covering homotopy property if for every space X, every map $\widetilde{f} : X \to E$ and every homotopy $F : X \times I \to B$ with $p \circ \widetilde{f} = F \circ J_0$, there exists a homtopy $\widetilde{F} : X \times I \to E$ such that $p \circ \widetilde{F} = F$ and $\widetilde{F} \circ J_0 = \widetilde{f}$, where $J_0 : X \to X \times I$ is $J_0(x) = (x, 0)$.

Here, we want to generalize covering homotopy property to homotopically covering homotopy property: **in the definition of the covering homotopy property, $p \circ \widetilde{F} = F$ is replace by $p \circ \widetilde{F} \simeq F$, rel $X \times \dot{I}$.**

A weak homotopical version of the covering homotopy property, that is, the weak covering homotopy property introduced by K. Fuchs [2]. A map $p : E \to B$ has the weak covering homotopy property if in the definition of the covering homotopy property, $\widetilde{F} \circ J_0 = \widetilde{f}$ is replace by the fiber homotopy $\widetilde{F} \circ J_0 \simeq_p \widetilde{f}$. A. Dold, et.al [1, 2, 4], studied the maps which have the weak covering homotopy property, called h-fibrations (or Dold fibrations). They proved that the weak covering homotopy property is invariant under the fiber homotopy equivalence, the fibers of an h-fibration have the same homotopy type, for every h-fibration there exists the long exact sequence, and etc (see [1, 2, 4]).

By using homotopically covering homotopy property, we introduce $\mathcal{H}$-fibrations as another homotopical generalization of fibrations.

1.1. **Preliminaries.** Throughout this article, all spaces are path connected, unless otherwise stated. A map $f : X \longrightarrow Y$ means a continuous function and $I := [0, 1]$. The map $\alpha : I \longrightarrow X$ is called a path from $x_0 = \alpha(0)$ to $x_1 = \alpha(1)$ and it's inverse α^{-1} defined by $\alpha^{-1}(t) = \alpha(1 - t)$. For two paths $\alpha, \beta : I \longrightarrow X$ with $\alpha(1) = \beta(0)$, $\alpha{*}\beta$ denotes the usual concatenation of the two paths. Also, all homotopies between paths is assumed to be relative to end points.

For given maps $p : E \to B$ and $f : X \to B$, a map $\widetilde{f} : X \to E$ is called a lifting of f if $p \circ \widetilde{f} = f$, and p has unique lifting property (ul), if every two lifts $\widetilde{f}, \overline{f}$ of f with the same image on some points of X, are equal. When $F : X \times I \longrightarrow Y$ is a map, we say that F is a homotopy from F_0 to F_1 and write $F : F_0 \simeq F_1$, where

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$F_i : X \longrightarrow Y$ is $F_i(x) = F(x,i)$, for $i = 0,1$. The constant map from X to Y which sends all points to $y \in Y$ is denoted by C_y.

2. Main results

In the definition of the covering homotopy property, diagrams commut in the TOP category(category of topological spaces and continuous maps). When we enter the HTOP category(category of topological spaces and homotopy class of maps), the commutativity of the diagrams means that the maps are homotopic.

Definition 2.1. *A map $p : E \to B$ is said has homotopically covering homotopy property, abbreviated by hchp, if for every space X, every map $\widetilde{f} : X \to E$ and every homotopy $F : X \times I \to B$ with $p \circ \widetilde{f} = F \circ J_0$, there exists a homotopy $\widetilde{F} : X \times I \to E$ such that $p \circ \widetilde{F} \simeq F$, rel $X \times \dot{I}$ and $\widetilde{F} \circ J_0 = \widetilde{f}$.*

Clearly every map with covering homotopy property has hchp, but the following example show that the converse is not necessarily true.

Example 2.2.
(i) Let $E = I \times I - \{(0, \frac{1}{2})\}$, $B = I$ and p be the projection on the first component. Moreover, let $F : X \times I \to B$, $\widetilde{f} : X \to E$ are maps with $p \circ \widetilde{f} = F \circ J_0$. Let $A = (1, \frac{1}{2})$, and define a homotopy $\widetilde{F} : X \times I \to E$ by

$$\widetilde{F}(x,t) = \begin{cases} \widetilde{f}(x) + 2(A - \widetilde{f}(x))t & t \in [0, \frac{1}{2}] \\ F(x,t) + 2(A - (F(x,t),0))(1-t) & t \in [\frac{1}{2}, 1]. \end{cases}$$

Define $H : X \times I \times I \to B$ by $H(x,t,s) = (1-s)p \circ \widetilde{F}(x,t) + sF(x,t)$. Then $H : p \circ \widetilde{F} \simeq F$ rel $X \times \dot{I}$ because $H(x,t,0) = p \circ \widetilde{F}(x,t)$, $H(x,t,1) = F(x,t)$ and for $i = 0,1$, $H(x,i,s) = (1-s)p \circ \widetilde{F}(x,i) + sF(x,i) = p \circ \widetilde{F}(x,i) = F(x,i)$. Also note that, $\widetilde{F} \circ J_0 = \widetilde{f}$.
(ii) Let $E = \{(t,0) | t \in I\} \cup \{(t,t) | t \in I - \{1\}\}$, $B = I$ and $p : E \to B$ be the projection on the first component. Let $\widetilde{f} : X \to E$ and $F : X \times I \to B$ be maps with $p \circ \widetilde{f} = F \circ J_0$. Define,

$$\widetilde{F}(x,t) = \begin{cases} \big((1-2t)pr_1\widetilde{f}(x), (1-2t)pr_2\widetilde{f}(x)\big) & t \in [0, \frac{1}{2}] \\ \big((2t-1)F(x,t), 0\big) & t \in [\frac{1}{2}, 1] \end{cases} .$$

Therefore, by the gluing lemma $\widetilde{F}$ is continuous. Define $H : X \times I \times I \to B$ by $H(x,t,s) = (1-s)p \circ \widetilde{F}(x,t) + sF(x,t)$. Then for every $x \in X$ and every $t, s \in I$ have

$$\begin{aligned} &H(x,t,0) = p \circ \widetilde{F}(x,t), \\ &H(x,t,1) = F(x,t), \\ &H(x,0,s) = (1-s)p \circ \widetilde{F}(x,0) + sF(x,0) = p \circ \widetilde{F}(x,0) = F(x,0), \\ &H(x,1,s) = (1-s)p \circ \widetilde{F}(x,1) + sF(x,1) = p \circ \widetilde{F}(x,1) = F(x,1). \end{aligned}$$

Moreover, $\widetilde{F} \circ J_0 = \widetilde{f}$.

A map $p : E \to B$ has path lifting property if for a given $b \in B$, $e \in p^{-1}(b)$ and a path α in B beginning at b, there exists a path $\widetilde{\alpha}$ in E such that $\widetilde{\alpha}(0) = e$ and $p \circ \widetilde{\alpha} = \alpha$, ([5]). Also by replacing $p \circ \widetilde{\alpha} = \alpha$ by $p \circ \widetilde{\alpha} \simeq \alpha$, rel $\dot{I}$, it is said that p has homotopically path lifting property and $\widetilde{\alpha}$ is a homotopically lifting of α, [6]. We know that fibrations and h-fibrations have path lifting property and homotopically path lifting property (see [6, 5]). Here, we prove that maps with hchp have homotopically path lifting property.

Proposition 2.3. *If $p : E \to B$ ihas hchp, then p has homotopically path lifting property.*

Proof. If α is a path in B and $e \in p^{-1}(\alpha(0))$, we show that α has a homotopically lifting at e. Let $F : \{*\} \times I \to B$ be the homotopy defined by $F(*, t) = \alpha(t)$ and $\widetilde{f} : \{*\} \to E$ be the map $\widetilde{f}(*) = e$. Then $p \circ \widetilde{f} = F \circ J_0$ and since p is an $\mathcal{H}$-fibration, there exist two homotopies $\widetilde{F} : \{*\} \times I \to E$ and $H : \{*\} \times I \times I \to B$ such that $H : p \circ \widetilde{F} \simeq F$ rel $\{*\} \times \dot{I}$ and $\widetilde{F} \circ J_0 = \widetilde{f}$. Let $\widetilde{\alpha}(t) = \widetilde{F}(*, t)$ and define $\overline{H} : I \times I \to B$ by $\overline{H}(s, t) = H(*, s, t)$. Therefore we have $\widetilde{\alpha}(0) = \widetilde{F}(*, 0) = \widetilde{F} \circ J_0(*) = \widetilde{f}(*) = e$ and $\overline{H} : p \circ \widetilde{\alpha} \simeq \alpha$ rel $\dot{I}$, because for every $s, t \in I$ have

$$\begin{aligned}
\overline{H}(s, 0) &= H(*, s, 0) = p \circ \widetilde{F}(*, s) = p \circ \widetilde{\alpha}(s), \\
\overline{H}(s, 1) &= H(*, s, 1) = F(*, s) = \alpha(s), \\
\overline{H}(0, t) &= H(*, 0, t) = p \circ \widetilde{F}(*, 0) = p \circ \widetilde{\alpha}(0) = F(*, 0) = \alpha(0), \\
\overline{H}(1, t) &= H(*, 1, t) = p \circ \widetilde{F}(*, 1) = p \circ \widetilde{\alpha}(1) = F(*, 1) = \alpha(1).
\end{aligned}$$

□

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INVARIANT INFINITE SERIES METRICS ON REDUCED Σ-SPACES

SIMIN ZOLFEGHARZADEH AND MEGERDICH TOOMANIAN

Abstract. In this paper we study the geometric properties of Finsler $\Sigma-$spaces. We prove that Infinite series $\Sigma-$spaces , are Riemannian.

Key words and phrases: Finsler metric; $(\alpha, \beta)-$ metric; infinite series metric.

1. Introduction

Let M be a C^∞ manifold and $\mu : M \times M \longrightarrow M$, $\mu(x, y) = x.y$ be a differentiable multiplication. The space M with the multiplication μ is said to be symmetric if the following conditions hold.

(1) $x.x = x$
(2) $x.(x.y) = y$
(3) $x.(y.z) = (x.y)(x.z)$
(4) Every point x has a neighborhood U such that $x.y = y$ implies $y = x$, for all $y \in U$.

The notion of symmetric spaces is due to E. Cartan and reformulated by O. Loos as pair (M, μ) with conditions $(1)-(4)$ in [14]. A. J. Ledger [12, 11] initiated the study later, generalized symmetric spaces or regular $s-$spaces. Let M be a $C^\infty-$manifold with a family of maps $\{s_x\}_{x\in M}$. The space M is said to be a regular $s-$space if the following conditions hold.

(a) $s_x x = x$,
(b) s_x is a diffeomorphism,
(c) $s_x \circ s_y = s_{s_x y} \circ s_x$,
(d) $(s_x)_*$ has only one fixed vector, the zero vector.

$\Sigma-$spaces and reduced $\Sigma-$spaces where first introduced by O. Loos [14] as generalisation of reflection spaces and symmetric spaces [13]. They include also the class of regular $s-$manifolds [7].

The definition of symmetric Finsler space is a natural generalization of E. Cartan's definition of Riemannian symmetric spaces. We call a Finsler space (M, F) as a symmetric Finsler space if for any point $p \in M$ there exists an involutive isometry s_p of (M, F) such that p is an isolated fixed point of s_p.

If we drop the involution property in the definition of symmetric Finsler space keeping the property $s_x \circ\ s_y = s_z \circ s_x,\ z = s_x(y)$ we get a bigger class of Finsler manifolds as symmetric Finsler spaces [4, 6, 8, 17]. Finsler $\Sigma-$spaces were first proposed and studied by the second authors in [9].

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2. Preliminaries

A Finsler metric on a C^∞ manifold of dimension n, is a function $F : TM \longrightarrow [0,\infty)$ which has the following properties.

(i) F is C^∞ on $TM_0 = TM\ \{0\}$,
(ii) F is positively 1-homogeneous on the fibers of tangent bundle TM,
(iii) For any non-zero $y \in T_xM$, the fundamental tensor $g_y : T_xM \times T_xM \longrightarrow R$ on T_xM is positive definite,

$$g_y(u,v) = \frac{1}{2}\frac{\partial^2}{\partial s\partial t}[F^2(y+su+tv)]|_{s=t=0}, \quad u,v \in T_xM.$$

Then (M,F) is called $n-$dimensional Finsler manifold.

One of the main quantities in Finsler geometry is the flag curvature which is defined as follows:

$$K(P,y) = \frac{g_y(R(u,y)y,u)}{g_y(y,y)g_y(u,u) - g_y^2(y,u)},$$

where $P = span\{u,y\}$ is a $2-$plane in T_xM, $R(u,y)y = \nabla_u\nabla_y y - \nabla_y\nabla_u y - \nabla_{[u,y]}y$ and ∇ is the Chern connection induced by F [16, 3]. For a Finsler metric F on $n-$dimensional manifold M, the Busemann-Hausdorff volume form $dV_F = \sigma_F(x)dx^1 \ldots dx^n$ is defined by

$$\sigma_F(x) = \frac{Vol(B^n(1))}{Vol\{(y^i) \in R^n | F(y^i \frac{\partial}{\partial x^i}|_x) < 1\}}.$$

Let $G^i := \frac{1}{4}g^{il}[\frac{\partial^2(F^2)}{\partial x^k\partial y^l}y^k - \frac{\partial(F^2)}{\partial x^l}]$, denote the geodesic coefficients of F in the same local coordinate system. The $S-$curvature can be defined by

$$S(y) = \frac{\partial G^i}{\partial y^i}(x,y) - y^i\frac{\partial}{\partial x^i}[ln\sigma_F(x)],$$

where $y = y^i\frac{\partial}{\partial x^i}|_x \in T_xM$ (see [3]). The Finsler metric F is said to be of isotropic $S-$curvature if

$$S = (n+1)cF,$$

where $c = c(x)$ is a scalar function on M.
Let (M,F) be an $n-$dimensional Finsler manifold. The non-Riemannian quantity $\Xi-$curvature $\Xi = \Xi_i dx^i$ on the tangent bundle TM is defined by

$$\Xi_i = S_{.i|m}y^m - S_{|i},$$

where S denotes the $S-$curvature, "." and "—" denote the vertical and horizontal covariant derivatives, respectively. We say that a Finsler metric have almost vanishing $\Xi-$curvature if $\Xi_i = -(n+1)F^2(\frac{\theta}{F})_{y^i}$ where $\theta = \theta_i(x)y^i$ is a 1-form on M [16, 5].

3. $(\alpha,\beta) - \Sigma$- spaces

We first recall the definition and some basic results concerning $\Sigma-$spaces [10].

Definition 3.1. *Let M be a smooth connected manifold, Σ a Lie group, and $\mu : M \times \Sigma \times M \longrightarrow M$ a smooth map. Then the triple (M,Σ,μ) is a $\Sigma-$space if it satisfies*

(Σ_1): $\mu(x,\sigma,x) = x$,
(Σ_2): $\mu(x,e,y) = y$,
(Σ_3): $\mu(x,\sigma,\mu(x,\tau,y)) = \mu(x,\sigma\tau,y)$
(Σ_5): $\mu(x,\sigma,\mu(y,\tau,z)) = \mu(\mu(x,\sigma,y),\sigma\tau\sigma^{-1},\mu(x,\sigma,z))$

where $x,y,z \in M$, $\sigma,\tau \in \Sigma$ and e is the identity element of Σ. The triple (M,Σ,μ) is usually dinoted by M.

For a fixed point $x \in M$ we define a map $\sigma_x : M \longrightarrow M$ by $\sigma_x(y) = \mu(x,\sigma,y)$ and a map $\sigma^x : M \longrightarrow M$ by $\sigma^x(y) = \sigma_y(x)$. with respect to these maps the above conditions became

($\Sigma_1^{'}$): $\sigma_x(x) = x$,
($\Sigma_2^{'}$): $e_x = id_M$,
($\Sigma_3^{'}$): $\sigma_x\tau_x = (\sigma\tau)_x$
($\Sigma_4^{'}$): $\sigma_x\tau_y\sigma_x^{-1} = (\sigma\tau\sigma^{-1})\sigma_x(y)$.

For each $x \in M$ by Σ_x we denote the image of Σ under the map $\Sigma \longrightarrow \Sigma_x$, $\sigma \longrightarrow \sigma_x$. For each $\sigma \in \Sigma$ we define (1,1)-tensor field S^σ on the $\Sigma-$space M by

$$S^\sigma X_x = (\sigma_x)_* X_x \qquad \forall x \in M, X_x \in T_xM.$$

Clearly S^σ is smooth.

Definition 3.2. *A $\Sigma-$space M is a reduced $\Sigma-$space if for each $x \in M$,*

(1) *T_xM is generated by the set of all $\sigma^x(X_x)$, that is*

$$T_xM = gen\big\{(I - S^\sigma)X_x | X_x \in T_xM, \sigma \in \Sigma\big\},$$

(2) *If $X_x \in T_xM$ and $\sigma^x X_x = 0$ for all $\sigma \in \Sigma$ then $X_x = 0$, and thus no non-zero vector in T_xM is fixed by all S^σ.*

Definition 3.3. *A Finsler $\Sigma-$space, denoted by (M,Σ,F) is a reduced $\Sigma-$space together with a Finsler metric F which is invariant under Σ_p for $p \in M$.*

Definition 3.4. *let $\alpha = \sqrt{\tilde{a}_{ij}(x)y^iy^j}$ be a norm induced by a Riemannian metric $\tilde{a}$ and $\beta(x,y) = b_i(x)y^i$ be a 1-form on an n-dimensional manifold M , and let*

$$\|\beta(x)\|_\alpha := \sqrt{\tilde{a}^{ij}b_i(x)b_j(x)}. \tag{3.1}$$

Now , the function F is defined by ,

$$F := \alpha\phi(s), \quad s = \frac{\beta}{\alpha}, \tag{3.2}$$

where $\phi = \phi(s)$ is a positive c^∞ function on $(-b_0, b_0)$ satisfying

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad | s |\leq b < b_0. \tag{3.3}$$

Then by lemma 1.1.2 of [3],F is a Finsler metric if $\|\beta(x)\|_\alpha < b_0$ for any $x \in M$. A Finsler metric in the form (3.2) is called an $(\alpha,\beta)-$ metric [1, 3]. A Finsler space having the Finsler function ,

$$F(x,y) = \frac{\beta^2(x,y)}{\beta(x,y) - \alpha(x,y)}, \tag{3.4}$$

is called a Finsler space with an infinite series(α,β) - metric.

now we present the main results

Lemma 3.5. *Let (M, Σ, F) be an infinite series $\Sigma-$ space with $F = \dfrac{\beta^2}{\beta - \alpha}$ defined by the Riemannian metric $\tilde{a}$ and the vector field X. Then $(M, \Sigma, \tilde{a})$ is a Riemannian $\Sigma-$space.*

Proof: Let σ_x be a diffeomorphism $\sigma_x : M \longrightarrow M$ defined by $\sigma_x(y) = \mu(x, \sigma, y)$. Then for $p \in M$ and for any $y \in T_pM$, we have $F(p, Y) = F(\sigma_x(p), d\sigma_x(Y))$. Applying equation (3.4) we get

$$\frac{\tilde{a}(X_p, y)^2}{\tilde{a}(X_p, y) - \sqrt{\tilde{a}(y, y)}} = \frac{\tilde{a}(X_{\sigma_x(p)}, d\sigma_x(y))^2}{\tilde{a}(X_{\sigma_x(p)}, d\sigma_x(y)) - \sqrt{\tilde{a}(d\sigma_x(y), d\sigma_x(y))}},$$

which implies

$$\tilde{a}(X_p, y)^2\tilde{a}(X_{\sigma_x(p)}, d\sigma_x(y)) - \tilde{a}(X_p, y)^2\sqrt{\tilde{a}(d\sigma_x(y), d\sigma_x(y))}$$
$$= \tilde{a}(X_{\sigma_x(p)}, d\sigma_x(y))^2\tilde{a}(X_p, y) - \tilde{a}(X_{\sigma_x(p)}, d\sigma_x(y))^2\sqrt{\tilde{a}(y, y)}. \tag{3.5}$$

Applying the above equation to $-Y$, we get

$$\tilde{a}(X_p, y)^2\tilde{a}(X_{\sigma_x(p)}, d\sigma_x(y)) + \tilde{a}(X_p, y)^2\sqrt{\tilde{a}(d\sigma_x(y), d\sigma_x(y))}$$
$$= \tilde{a}(X_{\sigma_x(p)}, d\sigma_x(y))^2\tilde{a}(X_p, y) + \tilde{a}(X_{\sigma_x(p)}, d\sigma_x(y))^2\sqrt{\tilde{a}(y, y)}, \tag{3.6}$$

Applying equations (3.5)and (3.6), we get

$$\tilde{a}(X_p, y) = \tilde{a}(X_{\sigma_x(p)}, d\sigma_x(y)), \tag{3.7}$$

subtracting equation (3.5)from equation (3.6) and using equation (3.7), we get

$$\tilde{a}(y, y) = \tilde{a}(d\sigma_x(y), d\sigma_x(y)).$$

Thus σ_x is an isometry with respect to the Riemannian metric $\tilde{a}$.□

Lemma 3.6. *Let $(M, \Sigma, \tilde{a})$ be a Riemannian $\Sigma-$space. Let F be an infinite series defined by the Riemannian metric $\tilde{a}$ and the vector field X. Then (M, Σ, F) is an infinite series $\Sigma-$space if and only if X is σ_x-invariant for all $x \in M$.*

Proof: Let X be σ_x-invariant. Then for any $p \in M$, we have $X_{\sigma_x(p)} = d\sigma_x X_p$. Then for any $y \in T_pM$ we have

$$\begin{aligned} F(\sigma_x(p), d\sigma_x y_p) =& \frac{\tilde{a}(X_{\sigma_x(p)}, d\sigma_x y_p)^2}{\tilde{a}(X_{\sigma_x(p)}, d\sigma_x y_p) - \sqrt{\tilde{a}(d\sigma_x y_p, d\sigma_x y_p)}} \\ =& \frac{\tilde{a}(d\sigma_x X_p, d\sigma_x y_p)^2}{\tilde{a}(d\sigma_x X_p, d\sigma_x y_p) - \sqrt{\tilde{a}(d\sigma_x y_p, d\sigma_x y_p)}} \\ =& \frac{\tilde{a}(X_p, y_p)^2}{\tilde{a}(X_p, y_p) - \sqrt{\tilde{a}(y_p, y_p)}} = F(p, y_p). \end{aligned}$$

conversely ,let F be a $\Sigma_M - invariant$, then for any $p \in M$ and $y \in T_pM$, we have $F(p, Y) = F(\sigma_x(p), d\sigma_x(Y))$.
Applying the lemma 3.5, we have $\tilde{a}(X_p, y) = \tilde{a}(X_{\sigma_x(p)}, d\sigma_x(y))$, which implies

$$\tilde{a}(y, y) = \tilde{a}(d\sigma_x(y), d\sigma_x(y)) \tag{3.8}$$

Combining the equation (3.7) and (3.8) , we get

$$\tilde{a}(X_x, y) = \tilde{a}(X_{\sigma_x(p)}, d\sigma_x(y)) \tag{3.9}$$

Therefore $d\sigma_x X_p = X_{\sigma_x(p)}$. □

Theorem 3.7. *An infinite series $\Sigma-$space must be Riemannian*

Proof: Let (M, Σ, F) be an infinet series $\Sigma-$space with $F = \dfrac{\beta^2}{\beta - \alpha}$ defined by the Riemannian metric $\tilde{a}$ and the vector field X. Let σ_x be a diffeomorphism defined by $\sigma_x(y) = \mu(x, \sigma, y)$. by lemma 3.5, $(M, \Sigma, \tilde{a})$ is a Riemannian $\Sigma-$space. Thus we have

$$\begin{aligned} F(x, d\sigma_x y) &= \frac{\tilde{a}(X_x, d\sigma_x(y))^2}{\tilde{a}(X_x, d\sigma_x(y)) - \sqrt{\tilde{a}(d\sigma_x(y), d\sigma_x(y))}} \\ &= \frac{\tilde{a}(X_x, d\sigma_x(y))^2}{\tilde{a}(X_x, d\sigma_x(y)) - \sqrt{\tilde{a}(y, y)}} = F(x, y). \end{aligned}$$

Therefore $\tilde{a}(X_x, d\sigma_x y) = \tilde{a}(X_x, y)$, $\forall y \in T_xM$. The tangent map $S^\sigma = (d\sigma_x)_x$ is an orthogonal transformation of T_xM without any nonzero fixed vectors. So we have $\tilde{a}(X_x, (S^\sigma - id)_x(y)) = 0$, $\forall y \in T_xM$. Since $(S - id)_x$ is an invertible linear transformation, we have $X_x = 0$, $\forall x \in M$. Hence F is Riemannian. $\square$

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G.R.C. OF EXPONENTIAL (α, β)-METRICS WITH ALMOST VANISHING Ξ-CURVATURE

MOSAYEB ZOHREHVAND

Abstract. In this paper, we study a class of Finsler metrics that is defined by a Riemannian metric α and a 1-form β on a manifold M. They are called (α, β)-metrics and have many applications in Physics, Biology, Control Theory and etc. We consider (α, β)-metric $F = \alpha(e^s + \epsilon s)$, $s := \frac{\beta}{\alpha}$ where $\epsilon \neq$ is a constant. It is called generalized Randers change (G.R.C.) exponential (α, β)-metric $\tilde{F} = \alpha e^s$. We prove that if F has almost vanishing Ξ-curvature then $\boldsymbol{\Xi} = 0$.

Key words and phrases: Ξ-curvature; exponential (α, β)-metric; Randers change.

1. Introduction

In Finsler geometry, there are several geometric quantities: Riemannian quantities including the Riemannian curvature, the flag curvature and non-Riemannian quantities including the distortion, the (mean) Cartan curvature, the S-curvature, the (mean) Berwald curvature and the mean Landsberg curvature, etc. They are vanishing for Riemannian metrics, hence they are said to be non-Riemannian [6].

S-curvature $\mathbf{S}$ is an importan non-Riemannian quantity in Finsler geometry which has impact to the flag curvature of a Finsler metric[3]. Using S-curvature, we can define the non-Riemannian quantity Ξ-curvature $\boldsymbol{\Xi} = \Xi_i dx^i$ as follow:

$$\Xi_i := \mathbf{S}_{.i|m} y^m - \mathbf{S}_{|i},$$

where . and $|$ denote the vertical and horizontal covariant derivative with respect to the Berwald connection of F, respectively. F is said to has almost vanishing Ξ-curvature, if there exists a 1-form $\theta = \theta_i dx^i$ on the manifold M such that

$$\Xi_i = -(n+1) F^2 \Big(\frac{\theta}{F}\Big)_{y^i}. \tag{1.1}$$

A rich and important class of Finsler metrics is (α, β)-metrics, which was first introduced by M. Matsumoto [5]. The simplest class of (α, β)-metrics are Randers metrics that have important applications in physics and mathematics [1]. It was first introduced and studied by G. Randers and is of the form $F = \alpha + \beta$ where α is a Riemannian metric and β is a 1-form [7].

Due to this issue, for a Finsler metric F, one can consider the change

$$F \to \tilde{F} := F + \epsilon\beta,$$

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where ϵ is a nonzero constant. This transformation is called generalized Randers change (G.R.C.) of F, because $\tilde{F}$ is reduced to a Randers metric when $F=\alpha$ is a Riemannian metric and $\epsilon = 1$.

The (α, β)-metric $F = \alpha \exp(s)$, $s := \beta/\alpha$, is called exponential metric and studied by many authors [8, 9, 11, 12]. This metric is interesting, because the exponential metric

$$F = \alpha \exp\Big(\int_0^s \frac{q\sqrt{b^2-t^2}}{1+qt\sqrt{b^2-t^2}}dt\Big),$$

is a almost regular unicorn metric, where $b := \|\beta\|_\alpha$ and q is a constant. A unicorn metric is a Landsberg metric that is not Berwaldian [10]. This paper is devoted to study of the generalized Randers change of exponential (α,β)-metric $F = \alpha\exp(s)$, $s := \beta/\alpha$ that has almost vanishing Ξ-curvature.

2. Preliminaries

For a Finsler space (M, F), The fundamental tensor $(\mathbf{g}_y) = (g_{ij}(x, y))$ of F is a quadratic form on T_xM that is defined

$$g_{ij}(x,y) := \frac{1}{2}[F^2]_{y^iy^j}(x,y).$$

The Finsler metric F induces a gelobal vector field $\mathbf{G}$ on the slight tangent bundle $TM_0 := TM - \{0\}$ that is given by

$$\mathbf{G} := y^i\frac{\partial}{\partial x^i} - 2G^i\frac{\partial}{\partial y^i},$$

on the standard induced coordinate (x^i, y^i). The coefficients $G^i := G^i(x,y)$ are local functions on TM_0 that are defined by

$$G^i := \frac{1}{4}g^{il}\Big\{[F^2]_{x^my^l}y^m - [F^2]_{x^l}\Big\}.$$

The Busemann-Hausdorff volume form $dV_F = \sigma_F(x)dx^i \wedge \ldots \wedge dx^n$ associated to Finsler metric F is defined by

$$\sigma_F(x) := Vol\Big(\frac{\mathbb{B}^n(1)}{Vol\Big\{(y^i)\in R^n \Big| F\big(y^i\frac{\partial}{\partial x^i}\big) < 1\Big\}}\Big).$$

The S-curvature $\mathbf{S}$ can be defined as follow:

$$\mathbf{S} := \frac{\partial G^i}{\partial x^i} - y^i\frac{\partial \ln \sigma_F}{\partial x^i}.$$

From S-curvature, we define the non-Riemannian quantity Ξ-curvature $\mathbf{\Xi} = \Xi_i dx^i$ on TM as follow:

$$\Xi_i := \mathbf{S}_{.i|m}y^m - \mathbf{S}_{|i}, \tag{2.1}$$

where . and | denote the vertical and horizontal covariant derivative with respect to the Berwald connection of F, respectively.

A Finsler metric F is an (α,β) -metric if $F = \alpha\phi(s)$, $s := \beta/\alpha$, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric, $\beta = b_i(x)y^i$ is a 1-form with $||\beta_x|| < b_0$, $x \in M$ and $\phi(s)$ is a positive C^∞ function on $(-b_0, b_0)$ satisfying

$$\phi(s) - s\phi^{'}(s) + (b^2 - s^2)\phi^{''}(s) > 0, \quad |s| \le b < b_0. \tag{2.2}$$

In this case, the metric $F = \alpha\phi(s)$ is a positive definite Finsler metric [4].

Let

$$r_{ij} := \frac{1}{2}(b_{i;j} + b_{j;i}), \quad s_{ij} := \frac{1}{2}(b_{i;j} - b_{j;i}),$$

where $b_{i;j}$ denote the coefficients of the covariant derivative of β with respect to α. It is easy to see that the 1-form β is closed if and only if $s_{ij} = 0$ and it is parallel with respect to α if and only if $r_{ij} = s_{ij} = 0$. Furthermore, we denote

$$\begin{aligned} r^i_{\ j} :=& a^{ik} r_{kj}, \quad r_{00} := r_{ij} y^i y^j, \\ r_{i0} :=& r_{ij} y^j, \quad r := r_{ij} b^i b^j, \\ s^i_{\ j} :=& a^{ik} s_{kj}, \quad s_{i0} := s_{ij} y^j, \\ s_i :=& b_j s^j_{\ i}, \quad s_0 := s_i y^i, \end{aligned}$$

where $b^i := a^{ij} b_j$.

The geodesic coefficients G^i of an (α,β)-metric $F = \alpha\phi(s)$ are given by [2]

$$G^i = G^i_\alpha + \alpha Q s^i_{\ 0} + \{-2Q\alpha s_0 + r_{00}\}\Big\{\Psi b^i + \Theta\alpha^{-1} y^i\Big\}, \tag{2.3}$$

where G^i_α is the geodesic coefficients of α and

$$Q = \frac{\phi^{'}}{\phi - s\phi^{'}},$$

$$\Theta = \frac{\phi\phi^{'} - s(\phi\phi^{''} + \phi^{'}\phi^{'})}{2\phi((\phi - s\phi^{'}) + (b^2 - s^2)\phi^{''})},$$

$$\Psi = \frac{\phi^{''}}{2((\phi - s\phi^{'}) + (b^2 - s^2)\phi^{''})}.$$

3. Main results

In this section, we consider (α,β)-metric $F = \alpha(e^s + \epsilon s)$, $s := \frac{\beta}{\alpha}$ with almost vanishing Ξ-curvature.

Tayebi and Amini in [9] obtained the formula of Ξ-curvature for an (α,β)-metric $F = \alpha\phi(s)$ as follow.

$$\Xi_i = H_{.i;m} y^m - H_{;i} - 2H_{.i.m} H^m, \tag{3.1}$$

where

$$H := (n+1)\alpha^{-1} A\Theta + Q' s_0 + \alpha^{-1} A\Psi'(b^2 - s^2) + 2\Psi[r_0 - sQs_0 - Q'(b^2 - s^2)s_0],$$

";" denotes the horizontal covariant derivative with respect to α and

$$A := r_{00} - 2\alpha Q s_0.$$

Theorem 3.1. *Let $F = \alpha(e^s + \epsilon s)$, $s := \beta/\alpha$ be an (α,β)-metric on an n-dimensional manifold M with $(n \geq 3)$, where $\epsilon \neq 0$ is a constant. Let F has almost vanishing Ξ-curvature, then $\boldsymbol{\Xi} = 0$.*

Proof. Let $F = \alpha\phi(s)$ has almost vanishing $\boldsymbol{\Xi}$-curvature, thus there exists an 1-form $\theta := \theta_i(x) y^i$ on M, such that

$$\Xi_i = -(n+1)F^2\Big(\frac{\theta}{F}\Big)_{y^i} = -(n+1)[\alpha\phi\theta_i - \frac{\phi}{\alpha}\theta y_i - \frac{\phi'}{\alpha}\theta(\alpha b_i - s y_i)]. \tag{3.2}$$

Putting $\phi(s) = e^s + \epsilon s$ and (3.2) in (3.1) and using maple program, we obtain

$$A_{i0} + A_{i1}e^s + A_{i2}e^{2s} + A_{i3}e^{3s} + A_{i4}e^{4s} = \lambda[B_{i0} + B_{i1}e^s + B_{i2}e^{2s} + B_{i3}e^{3s} + B_{i4}e^{4s} + B_{i5}e^{5s} + B_{i6}e^{6s}], \tag{3.3}$$

where $A_{i4}, \ldots, A_{i0}$ and $B_{i6}, \ldots, B_{i0}$ are polynomials of s and b and

$$\lambda = 2(n+1)\alpha^4(1 - s + b^2 - s^2)^5(s-1)^4.$$

From (3.3), we obtain

$$B_{i5} = B_{i6} = 0. \tag{3.4}$$

Since $B_{i5} = \epsilon[s\alpha\theta_i - \frac{s}{\alpha}\theta y_i - \frac{\theta}{\alpha}(\alpha b_i - sy_i)]$ and $B_{i6} = \alpha\theta_i - \frac{\theta}{\alpha}y_i - \frac{\theta}{\alpha}(\alpha b_i - sy_i)$, thus

$$\frac{1}{\alpha^2}[\alpha^3\theta_i - \alpha\theta y_i - \theta(\alpha^2 b_i - \beta y_i)] = 0 \tag{3.5}$$

Contracting (3.5) with b^i, we get

$$\alpha^3\theta_i b^i - \alpha\beta\theta - \theta(\alpha^2 b^2 - \beta^2) = 0. \tag{3.6}$$

From (3.6), we have

$$\alpha^3\theta_i b^i - \alpha\beta\theta = \theta(\alpha^2 b^2 - \beta^2). \tag{3.7}$$

We see that the left hand side of (3.7) is irrational and the right hand side is rational. On the other hand $b^2\alpha^2 - \beta^2 \neq 0$, thus from (3.7), we conclude $\theta = 0$, i.e. $\boldsymbol{\Xi} = 0$. $\square$

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GEODESIC VECTORS OF EXPONENTIAL METRIC ON FOUR DIMENSIONAL LIE GROUPS

MILAD ZEINALI LAKI AND DARIUSH LATIFI

Abstract. In this paper, we consider invariant exponential metrics and describe all geodesic vectors and investigate the set of all homogeneous geodesics on left invariant hypercomplex 4-dimensional simply connected Lie groups.

Key words and phrases: Complex structure; Exponential metric; Geodesic vector; Homogeneous geodesic.

1. Introduction

A Finsler manifold is a manifold M where each tangent space is equipped with a Minkowski norm, that is, a norm that is not necessarily induced by an inner product (here, a Minkowski norm has no relation to indefinite inner products). This norm also induces a canonical inner product. Finsler geometry is named after Paul Finsler who studied it in his doctoral thesis in 1917.

The important family of Finsler metrics is the (α, β)- metrics. The notion of (α, β)- metrics are introduced by Matsumoto [6]. If $F = \alpha + \beta$, then we get the Randers metric. This metric is an (α, β)- metric that introduced by Ingarden. An (α, β)- metric is a Finsler metric of the form $F = \alpha\varphi(s)$, $s = \frac{\beta}{\alpha}$ where $\alpha = \sqrt{\tilde{a}_{ij}(x)y^i y^j}$ is induced by a Riemannian metric $\tilde{a} = \tilde{a}_{ij}dx^i \otimes dx^j$ on a connected smooth n- dimensional manifold M and $\beta = b_i(x)y^i$ is a 1- form on M. We note that, the important kinds of (α, β)- metrics are Kropina metric $F = \alpha^2/\beta$, square metric $F = (\alpha+\beta)^2/\alpha$, exponential metric $F = \alpha \exp(\beta/\alpha)$ and Matsumoto metric $F = \alpha^2/(\alpha - \beta)$.

The important concepts in Finsler geometry is geodesics. Geodesics in a manifold is the generalization of concept of a straight line in an Euclidean space. A geodesic in a homogeneous Finsler space $(G/H, F)$ is called homogeneous geodesic if it is an orbit of a one-parameter subgroup of G. Homogeneous geodesics on homogeneous Riemannian manifolds have been studied by many authors. Latifi has extended the concept of homogeneous geodesics in homogeneous Finsler spaces [5].

suppose (M, F) be a connected homogeneous Finsler space, G is a connected transitive group of isometries of M and H is the isotropy subgroup at a point $o \in M$. Therefore, M is naturally identified with the coset space G/H with G-invariant Finsler metric F. Also, in this case the Lie algebra $\mathfrak{g}$ of G has a reductive decomposition

$$\mathfrak{g} = \mathfrak{m} + \mathfrak{h},$$

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where $\mathfrak{m} \subset \mathfrak{g}$ is a subspace of $\mathfrak{g}$ isomorphic to the T_oM and $\mathfrak{h}$ is the Lie algebra of H. In this paper we study homogeneous geodesics of left invariant Exponential metrics on left invariant hypercomplex 4-dimensional simply connected Lie groups.

2. Preliminaries

Definition 2.1. *Let M be a smooth n- dimensional C^∞ manifold and TM be its tangent bundle. A Finsler metric on a manifold M is a non-negative function $F : TM \to \mathbb{R}$ with the following properties [1].*

1. *F is smooth on the slit tangent bundle $TM^0 := TM\backslash\{0\}$.*
2. *$F(x, \lambda y) = \lambda F(x, y)$ for any $x \in M$, $y \in T_xM$ and $\lambda > 0$.*
3. *The following bilinear symmetric form $g_y : T_xM \times T_xM \to \mathbb{R}$ is positive definite*

$$g_y(u, v) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t}F^2(x, y + su + tv)|_{s=t=0}.$$

Let $\alpha = \sqrt{\widetilde{a}_{ij}\,(x)\,y^i y^j}$ be a norm induced by a Riemannian metric $\widetilde{a}$ and $\beta\,(x, y) = b_i(x)y^i$ be a 1-form on an n- dimensional manifold M. Let

$$b := \|\beta(x)\|_\alpha := \sqrt{\widetilde{a}(x)b_i(x)b_j(x)}.$$

Now, let the function F is defined as follows

$$F := \alpha\varphi(s), \quad s = \frac{\beta}{\alpha}, \tag{2.1}$$

where $\varphi = \varphi(s)$ is a positive C^∞ function on $(-b_0, b_0)$ satisfying

$$\varphi\,(s) - s\varphi'\,(s) + \left(b^2 - s^2\right)\varphi''\,(s) > 0, \qquad |s| \leq\; b < b_0.$$

Then F is a Finsler metric if $\|\beta(x)\|_\alpha < b_0$ for any $x \in M$. A Finsler metric in the form (2.1) is called an (α, β)- metric [8].

A Finsler space having the Finsler function:

$$F\,(x, y) = \alpha(x, y)\exp(\frac{\beta(x, y)}{\alpha(x, y)}),$$

is called a exponential space. We note that the Riemannian metric $\tilde{a}$ induces an inner product on any cotangent space T_x^*M such that $\langle dx^i(x), dx^j(x)\rangle = \tilde{a}^{ij}(x)$. The induced inner product on T_x^*M induces a linear isomorphism between T_x^*M and T_xM. Then the 1-form β corresponds to a vector field $\tilde{X}$ on M such that

$$\tilde{a}(y, \tilde{X}(x)) = \beta(x, y). \tag{2.2}$$

Also we have $\|\beta(x)\|_\alpha = \|\tilde{X}(x)\|_\alpha$. Therefore we can write exponential metric as follows:

$$F(x, y) = \sqrt{\tilde{a}(y, y)}\exp(\frac{\tilde{a}(X, y)}{\sqrt{\tilde{a}(y, y)}}). \tag{2.3}$$

Now consider the Chern connection on π^*TM whose coefficients are denoted by Γ^i_{jk}. Let $\gamma(t)$ be a smooth regular curve in M with velocity field V. Suppose $W(t) := W^i(t)\frac{\partial}{\partial x^i}$ be a vector field along γ. Then the covariant derivative $D_V W$ with reference vector V have the form

$$\Big[\frac{dW^i}{dt} + W^j V^k(\Gamma^i_{jk})_{(\gamma,V)}\Big]\frac{\partial}{\partial x^i}|_{\gamma(t)}.$$

A curve $\gamma(t)$ with the velocity $V = \dot{\gamma}(t)$, is a Finslerian geodesic if

$$D_V\Big[\frac{V}{F(V)}\Big] = 0, \quad \text{with reference vector} \quad V.$$

Definition 2.2. *Suppose $(G/H, F)$ be a homogeneous Finsler manifold with a fixed origin o. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebra of G and H respectively and $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ a reductive decomposition. Therefore, a homogeneous geodesic through the $o \in G/H$ is a geodesic $\gamma(t)$ of the form*

$$\gamma(t) = \exp(tZ)(o), \quad t \in \mathbb{R}, \tag{2.4}$$

where Z is a nonzero vector of $\mathfrak{g}$.

In Riemannian setting the authors in [4], proved that a $X \in \mathfrak{g} - \{0\}$ is a geodesic vector if and only if

$$\langle [X, Y]_{\mathfrak{m}}, X_{\mathfrak{m}} \rangle = 0, \quad \forall Y \in \mathfrak{m}. \tag{2.5}$$

After this, the second author in Finsler setting shown that

Lemma 2.3 ([5])**.** *Suppose $(G/H, F)$ be a homogeneous Finsler space with a reductive decomposition*

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m}.$$

Therefore, $Y \in \mathfrak{g} - \{0\}$ is a geodesic vector if and only if

$$g_{Y_{\mathfrak{m}}}(Y_{\mathfrak{m}}, [Y, Z]_{\mathfrak{m}}) = 0, \quad \forall Z \in \mathfrak{m}, \tag{2.6}$$

where the subscript $\mathfrak{m}$ indicates the projection of a vector from $\mathfrak{g}$ to $\mathfrak{m}$.

3. Geodesic Vectors of Exponential metric On Four Dimensional Lie Group

An almost complex structure on a real differentiable manifold M is a tensor field J which is, at every point x of M, an endomorphism of the tangent space T_xM such that $J^2 = -1$, where 1 denotes the identity transformation of T_xM. Note that for any two vector fields X and Y, we define the Nijenhuis tensor N as

$$N(X, Y) = [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY]. \tag{3.1}$$

A hypercomplex manifold is a manifold M with three globally-defined, integrable complex structures I, J, K satisfying the quaternion identities

$$I^2 = J^2 = K^2 = -1, \quad \text{and} \quad IJ = K = -JI. \tag{3.2}$$

Obata [7] proved that a hypercomplex manifold admits a unique torsion-free connection ∇ such that

$$\nabla I = \nabla J = \nabla K = 0.$$

Now let M be a 4-dimensional manifold. A hypercomplex structure on M is a family $\mathbb{H} = \{J_\alpha\}_{\alpha=1,2,3}$ of fiberwise endomorphism of TM such that

$$-J_2J_1 = J_1J_2 = J_3, \quad J_\alpha^2 = -Id_{TM}, \quad \alpha = 1, 2, 3, \tag{3.3}$$

$$N_\alpha = 0, \quad \alpha = 1, 2, 3, \tag{3.4}$$

where N_α is the Nijenhuis tensor (torsion) corresponding to J_α.

We note that, an almost complex structure is a complex structure if and only if it has no torsion [3]. Then the complex structures J_α, $\alpha = 1, 2, 3$, on a 4-dimensional manifold M form a hypercomplex if they satisfy in the relation (3.3).

Definition 3.1. *A Riemannian metric $\tilde{a}$ on a hypercomplex manifold $(M, \mathbb{H})$ is called hyper-Hermitian if for all vector fields X and Y on M and for all $\alpha = 1, 2, 3$ we have*

$$\tilde{a}(J_\alpha X, J_\alpha Y) = \tilde{a}(X, Y).$$

Definition 3.2. *A hypercomplex structure $\mathbb{H} = \{J_\alpha\}_{\alpha=1,2,3}$ on a Lie group G is said to be left invariant if for any $t \in G$ we have*

$$J_\alpha = Tl_t \circ J_\alpha \circ Tlt^{-1},$$

where Tl_t is the differential function of the left translation l_t.

In this section, we consider left invariant hyper-Hermitian Riemannian metrics on left invariant hypercomplex 4-dimensional simply connected Lie groups. Barberis shown that in this spaces, $\mathfrak{g}$ is either Abelian or isomorphic to one of the following Lie algebras.

$$[e_2, e_3] = e_4, \quad [e_3, e_4] = e_2, \quad [e_4, e_2] = e_3, \quad e_1 : \text{central}, \tag{3.5}$$

$$[e_1, e_3] = e_1, \quad [e_2, e_3] = e_2, \quad [e_1, e_4] = e_2, \quad [e_2, e_4] = -e_1, \tag{3.6}$$

$$[e_1, e_2] = e_2, \quad [e_1, e_3] = e_3, \quad [e_1, e_4] = e_4, \tag{3.7}$$

$$[e_1, e_2] = e_2, \quad [e_1, e_3] = \frac{1}{2}e_2, \quad [e_1, e_4] = \frac{1}{2}e_4, \quad [e_3, e_4] = \frac{1}{2}e_2, \tag{3.8}$$

where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis.

Now we want to describe all geodesics vectors of left invariant exponential metrics F defined by relation

$$F(x, y) = \sqrt{\tilde{a}(y, y)} \exp\Big(\frac{\tilde{a}(X, y)}{\sqrt{\tilde{a}(y, y)}}\Big).$$

By using the formula

$$g_y(u, v) = \frac{1}{2} \frac{\partial^2}{\partial s \partial t} F^2(x, y + su + tv)|_{s=t=0},$$

and some computations we get

$$\begin{aligned} g_y(u, v) = &\exp\Big(\frac{2\tilde{a}(X, y)}{\sqrt{\tilde{a}(y, y)}}\Big)\Big(\tilde{a}(u, v) + 2\tilde{a}(X, u)\tilde{a}(X, v) - \frac{\tilde{a}(X, y)\tilde{a}(y, u)\tilde{a}(y, v)}{\tilde{a}(y, y)^{3/2}}\Big) \\ &+ \exp\Big(\frac{2\tilde{a}(X, y)}{\sqrt{\tilde{a}(y, y)}}\Big)\frac{1}{\sqrt{\tilde{a}(y, y)}}\Big(\tilde{a}(X, u)\tilde{a}(y, v) + \tilde{a}(X, v)\tilde{a}(y, u) - \tilde{a}(X, y)\tilde{a}(u, v)\Big) \\ &+ \exp\Big(\frac{2\tilde{a}(X, y)}{\tilde{a}(y, y)}\Big)\frac{2\tilde{a}(X, y)}{\tilde{a}(y, y)}\Big(\frac{\tilde{a}(X, y)\tilde{a}(y, u)\tilde{a}(y, v)}{\tilde{a}(y, y)} - \tilde{a}(y, u)\tilde{a}(X, v) - \tilde{a}(X, v)\tilde{a}(y, v)\Big). \end{aligned}$$

Therefore, for all $z \in \mathfrak{g}$ we have

$$g_y(y, [y, z]) = \tilde{a}\Big(X + \Big(\frac{\sqrt{\tilde{a}(y, y)} - \tilde{a}(X, y)}{\tilde{a}(y, y)}\Big)y, [y, z]\Big)\sqrt{\tilde{a}(y, y)} \exp\Big(\frac{2\tilde{a}(X, y)}{\sqrt{\tilde{a}(y, y)}}\Big). \tag{3.9}$$

Now, by using lemma 2.3 and equation (3.9) a vector $y = \sum_{i=1}^4 y_i e_i$ of $\mathfrak{g}$ is a geodesic vector if and only if for each $j = 1, 2, 3, 4$,

$$\tilde{a}\Big(\sum_{i=1}^4 x_i e_i + \Big(\frac{\sqrt{\sum_{i=1}^4 y_i^2} - \sum_{i=1}^4 x_i y_i}{\sum_{i=1}^4 y_i^2}\Big)\sum_{i=1}^4 y_i e_i, [\sum_{i=1}^4 y_i e_i, e_j]\Big) = 0. \tag{3.10}$$

So, we get the following cases:

3.1. **Case (1).**

$$\begin{cases} j=2 \;\to\; & x_3y_4 - x_4y_3 = 0, \\ j=3 \;\to\; & x_4y_2 - x_2y_4 = 0, \\ j=4 \;\to\; & x_2y_3 - x_3y_2 = 0. \end{cases}$$

As a special case, if $X = x_1e_1$, then a vector y of G is a geodesic vector if and only if $y \in Span\{e_1\}$.

Corollary 3.3. *Let (M, F) be a Finsler space with exponential metric F defined by an invariant metric $\tilde{a}$ and an invariant vector eld $X = \sum_{i=1}^{4} x_ie_i$ on left invariant hypercomplex 4-dimensional simply connected Lie group and let (3.5) holds. Then geodesic vectors depending on x_2, x_3 and x_4.*

Theorem 3.4. *Let (M, F) be a Finsler space with exponential metric F defined by an invariant metric $\tilde{a}$ and an invariant vector eld $X = x_1e_1$ on left invariant hypercomplex 4-dimensional simply connected Lie group and let (3.5) holds. Then $y \in \mathfrak{g}$ is a geodesic vector of (M, F) if and only if y is a geodesic vector of $(M, \tilde{a})$.*

Proof. Let $y \in \sum_{i=1}^{4} y_ie_i \in \mathfrak{g}$. Let y is a geodesic vector of $(M, \tilde{a})$. By using (2.5) we have $\tilde{a}(y, [y, e_i]) = 0$ for each $i = 1, 2, 3, 4$. Therefore by using (3.10), y is a geodesic of (M, F).
Conversely, let $y = \sum_{i=1}^{5} y_ie_i \in \mathfrak{g}$ is a geodesic vector of (M, F), because $\tilde{a}(X, [y, e_i]) = 0$ for each $i = 1, 2, 3, 4$, by using (3.10) we have $\tilde{a}(y, [y, e_i]) = 0$. □

3.2. **Case (2).**

$$\begin{cases} j=1 \;\to\; & x_1y_3 + \frac{\sqrt{\sum y_i^2} - \sum x_iy_i}{\sum y_i^2}y_1y_3 + x_2y_4 + \frac{\sqrt{\sum y_i^2} - \sum x_iy_i}{\sum y_i^2}y_2y_4 = 0, \\ j=2 \;\to\; & x_1y_4 + \frac{\sqrt{\sum y_i^2} - \sum x_iy_i}{\sum y_i^2}y_1y_4 - (x_2y_3 + \frac{\sqrt{\sum y_i^2} - \sum x_iy_i}{\sum y_i^2}y_2y_3) = 0, \\ j=3 \;\to\; & x_1y_1 + \frac{\sqrt{\sum y_i^2} - \sum x_iy_i}{\sum y_i^2}y_1^2 + x_2y_2 + \frac{\sqrt{\sum y_i^2} - \sum x_iy_i}{\sum y_i^2}y_2^2 = 0, \\ j=4 \;\to\; & x_2y_1 - x_1y_2 = 0. \end{cases}$$

As a special case, if $X = x_3e_3 + x_4e_4$, then a vector y of G is a geodesic vector if and only if $y \in Span\{e_3, e_4\}$.

Corollary 3.5. *Let (M, F) be a Finsler space with exponential metric F defined by an invariant metric $\tilde{a}$ and an invariant vector eld $X = \sum_{i=1}^{4} x_ie_i$ on left invariant hypercomplex 4-dimensional simply connected Lie group and let (3.6) holds. Then geodesic vectors depending on x_1 and x_2.*

Theorem 3.6. *Let (M, F) be a Finsler space with exponential metric F defined by an invariant metric $\tilde{a}$ and an invariant vector eld $X = x_3e_3 + x_4e_4$ on left invariant hypercomplex 4-dimensional simply connected Lie group and let (3.6) holds. Then $y \in \mathfrak{g}$ is a geodesic vector of (M, F) if and only if y is a geodesic vector of $(M, \tilde{a})$.*

Proof. The proof is the same as before. □

3.3. **Case (3).**

$$\begin{cases} j=1 \ \rightarrow \quad x_2y_2+x_3y_3+x_4y_4+\frac{\sqrt{\sum y_i^2}-\sum x_iy_i}{\sum y_i^2}(y_2^2+y_3^2+y_4^2)=0,\\ j=2 \ \rightarrow \quad x_2y_1+\frac{\sqrt{\sum y_i^2}-\sum x_iy_i}{\sum y_i^2}y_2y_1=0,\\ j=3 \ \rightarrow \quad x_3y_1+\frac{\sqrt{\sum y_i^2}-\sum x_iy_i}{\sum y_i^2}y_3y_1=0,\\ j=4 \ \rightarrow \quad x_4y_1+\frac{\sqrt{\sum y_i^2}-\sum x_iy_i}{\sum y_i^2}y_4y_1=0. \end{cases}$$

As a special case, if $X = x_1e_1$, then a vector y of G is a geodesic vector if and only if $y \in Span\{e_1\}$.

Corollary 3.7. *Let (M, F) be a Finsler space with exponential metric F defined by an invariant metric $\tilde{a}$ and an invariant vector eld $X = \sum_{i=1}^{4} x_ie_i$ on left invariant hypercomplex 4-dimensional simply connected Lie group and let (3.7) holds. Then geodesic vectors depending on x_2, x_3 and x_4.*

Theorem 3.8. *Let (M, F) be a Finsler space with exponential metric F defined by an invariant metric $\tilde{a}$ and an invariant vector eld $X = x_1e_1$ on left invariant hypercomplex 4-dimensional simply connected Lie group and let (3.7) holds. Then $y \in \mathfrak{g}$ is a geodesic vector of (M, F) if and only if y is a geodesic vector of $(M, \tilde{a})$.*

Proof. The proof is the same as before. □

3.4. **Case (4).**

$$\begin{cases} j=1 \ \rightarrow \quad 2x_2y_2+x_3y_3+x_4y_4+\frac{\sqrt{\sum y_i^2}-\sum x_iy_i}{\sum y_i^2}(2y_2^2+y_3^2+y_4^2)=0,\\ j=2 \ \rightarrow \quad x_2y_1+\frac{\sqrt{\sum y_i^2}-\sum x_iy_i}{\sum y_i^2}y_2y_1=0,\\ j=3 \ \rightarrow \quad x_3y_1-x_2y_4+\frac{\sqrt{\sum y_i^2}-\sum x_iy_i}{\sum y_i^2}(y_3y_1-y_2y_4)=0,\\ j=4 \ \rightarrow \quad x_2y_3+x_4y_1+\frac{\sqrt{\sum y_i^2}-\sum x_iy_i}{\sum y_i^2}(y_4y_1+y_2y_3)=0. \end{cases}$$

As a special case, if $X = x_1e_1$, then a vector y of G is a geodesic vector if and only if $y \in Span\{e_1\}$.

Corollary 3.9. *Let (M, F) be a Finsler space with exponential metric F defined by an invariant metric $\tilde{a}$ and an invariant vector eld $X = \sum_{i=1}^{4} x_ie_i$ on left invariant hypercomplex 4-dimensional simply connected Lie group and let (3.8) holds. Then geodesic vectors depending on x_2, x_3 and x_4.*

Theorem 3.10. *Let (M, F) be a Finsler space with exponential metric F defined by an invariant metric $\tilde{a}$ and an invariant vector eld $X = x_1e_1$ on left invariant hypercomplex 4-dimensional simply connected Lie group and let (3.8) holds. Then $y \in \mathfrak{g}$ is a geodesic vector of (M, F) if and only if y is a geodesic vector of $(M, \tilde{a})$.*

Proof. The proof is the same as before. □

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QUARTER SYMMETRIC METRIC CONNECTIONS ON COSYMPLECTIC STATISTICAL MANIFOLDS

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Abstract. In this paper we define a quarter symmetric metric connection on cosymplectic statistical manifolds and study the geometry of these manifolds and their submanifolds. Also, we prove the induced connection on a submanifold with respect to a quarter symmetric metric connection is a querter symmetric metric connection and the second fundamental form coincides with the second fundamental form of the Levi-Civita connection.

Key words and phrases: statistical manifold; quarter symmetric; cosymplectic manifold.

1. Introduction

Definition 1.1. *Let $(\bar{M}, g)$ be a Riemannian manifold with Levi-Civita connection $\hat{\nabla}$. A pair $(\bar{\nabla}, g)$ is called a statistical structure on $\bar{M}$ if $\bar{\nabla}$ is an affine and torsion free connection and for all $X, Y, Z \in \mathcal{T}(\bar{M})$ we have* [1]

$$(\bar{\nabla}_X g)(Y, Z) = (\bar{\nabla}_Y g)(X, Z). \tag{1.1}$$

Then $(\bar{M}, g, \bar{\nabla})$ is said to be a statistical manifold.

An affine connection $\bar{\nabla}^*$ is called a dual connection of $\bar{\nabla}$ if

$$Xg(Y, Z) = g(\bar{\nabla}^*_X Y, Z) + g(Y, \bar{\nabla}_X Z). \tag{1.2}$$

$\bar{\nabla}^*$ satisfies in equation (1.1) and $(\bar{\nabla}^*)^* = \bar{\nabla}$. From compability of $\hat{\nabla}$ with g and (1.2) we obtain $\hat{\nabla} = \dfrac{1}{2}(\bar{\nabla} + \bar{\nabla}^*)$.

by defining $\bar{K}_X Y = \bar{\nabla}_X Y - \hat{\nabla}_X Y$. [5] $\bar{K}$ is a symmetric (1,2)-tensor field on $\bar{M}$, that is, $\bar{K}_X Y = \bar{K}_Y X$ and

$$g(\bar{K}_X Y, Z) = g(\bar{K}_X Z, Y). \tag{1.3}$$

Let $(\bar{M}, g)$ be a $(2n+1)$-dimensional Riemannian manifold. If there exist a (1,1)-tensor field φ, a structure vector field ξ and a 1-form η on M such that for all $X, Y \in \mathcal{T}(\bar{M})$

$$\varphi^2 = -I + \eta \otimes \xi, \tag{1.4}$$

$$\varphi(\xi) = 0, \quad \eta(\xi) = 1, \tag{1.5}$$

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(\varphi X, Y) = -g(X, \varphi Y), \tag{1.6}$$

then $(\bar{M}, \varphi, \xi, \eta, g)$ is said to be an almost contact metric manifold. An almost contact metric manifold $(\bar{M}, \varphi, \xi, \eta, g)$ is called a cosymplectic manifold if $(\hat{\nabla}_X \varphi)Y = 0$. Now, let M be a submanifold of statistical manifold $\bar{M}$, then the Gauss formulas

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for submanifold M of $\bar{M}$ with respect to statistical connections $\bar{\nabla}$ and $\bar{\nabla}^*$ are given by [2]

$$\bar{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{1.7}$$

$$\bar{\nabla}^*_X Y = \nabla^*_X Y + h^*(X,Y), \tag{1.8}$$

for all $X,Y \in \mathcal{T}(M)$, where ∇, ∇^* and h, h^* are induced statistical connections and second fundamental forms on M, respectively. Also M is called φ-invariant if $\varphi X \in \mathcal{T}(M)$, for all $X \in \mathcal{T}(M)$.

Theorem 1.2 ([4])**.** *Let $(\bar{M}, g, \varphi, \bar{\nabla}, \bar{\nabla}^*)$ be an almost contact statistical manifold. Then $(\bar{M}, g, \varphi, \bar{\nabla}, \bar{\nabla}^*)$ be a cosymplectic statistical manifold if and only if*

$$\bar{\nabla}_X \varphi Y - \varphi \bar{\nabla}^*_X Y = \bar{K}_X \varphi Y + \varphi \bar{K}_X Y. \tag{1.9}$$

Definition 1.3. *A linear connection $\tilde{\nabla}$ on a cosymplectic statistical manifold $(\bar{M}, g, \varphi, \eta, \xi, \bar{\nabla}, \bar{\nabla}^*)$ is said to be a quarter symmetric connection if its torsion tensor $\tilde{T}$ satisfies*

$$\tilde{T}(X,Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y] = \eta(Y)\varphi X - \eta(X)\varphi Y, \quad \forall X,Y \in \mathcal{T}(\bar{M}) \tag{1.10}$$

Moreover, if the quarter symmetric connection $\tilde{\nabla}$ satisfies $\tilde{\nabla} g = 0$, then $\tilde{\nabla}$ is called a quarter symmetric metric connection.

Now, for all $X,Y \in \mathcal{T}(\bar{M})$ we set

$$\tilde{\nabla}_X Y = \bar{\nabla}_X Y - \eta(X)\varphi Y - K_X Y, \quad \tilde{\nabla}_X Y = \bar{\nabla}^*_X Y - \eta(X)\varphi Y + K_X Y \tag{1.11}$$

It is easy to see that the torsion tensor $\tilde{T}$ with respect to linear connection $\tilde{\nabla}$ satisfies in (1.10).

In [3] the outhors in generel define a semi-symmetric metric connection on statistical manifolds and study the geometry of these manifolds and their submanifolds. In this paper we define a quarter symmetric metric connection on cosymplectic statistical manifolds. We prove for a quarter symmetric metric connection on cosymplectic statistical manifolds the relation $\tilde{\nabla}\varphi = 0$ holds.

2. Main results

Theorem 2.1. *Let $(\bar{M}, \bar{\nabla}, g)$ be a cosymplectic statistical manifold admitting a quarter symmetric linear connection $\tilde{\nabla}$ defined in (1.11). Then $\tilde{\nabla}$ is a metric connection.*

Proof. By using (1.1), (1.3), (1.6), (1.11) and Theorem 1.2 we obtain

$$(\tilde{\nabla}_X g)(Y,Z) = 0, \quad \forall X,Y,Z \in \mathcal{T}(\bar{M}).$$

It gives the assertion. □

Now, we prove any quarter symmetric metric connection on a cosymplectic statistical manifold satisfies in (1.11).

Theorem 2.2. *Let $(\bar{M}, \bar{\nabla}, g)$ be a cosymplectic statistical manifold admitting a quarter symmetric linear connection $\tilde{\nabla}$. Then $\tilde{\nabla}$ satisfies in (1.11).*

Proof. Let $\tilde{\nabla}$ be a metric connection satisfying (1.11) on cosymplectic statistical manifold $\bar{M}$ defined by

$$\tilde{\nabla}_X Y = \bar{\nabla}_X Y + H(X,Y), \tag{2.1}$$

where H is a $(1,2)$-tensor field on $\bar{M}$. From (1.1) and (2.1) we obtain

$$\begin{aligned} 0 = (\tilde{\nabla}_X g)(Y,Z) &= Xg(Y,Z) - g(\tilde{\nabla}_X Y, Z) - g(Y, \tilde{\nabla}_X Z) = Xg(Y,Z) \\ &- g(\bar{\nabla}_X Y + H(X,Y), Z) - g(Y, \bar{\nabla}_X Z + H(X,Z)) \\ &= -2g(K_X Z, Y) - g(H(X,Y), Z) - g(H(X,Z), Y). \end{aligned}$$

So

$$g(H(X,Y),Z) + g(H(X,Z),Y) = -2g(K_X Z, Y).$$

Now, from (2.1) we have

$$\tilde{T}(X,Y) = H(X,Y) - H(Y,X).$$

By using (1.10) we obtain

$$\begin{aligned} g(\tilde{T}(X,Y),Z) + g(\tilde{T}(Z,X),Y) + g(\tilde{T}(Z,Y),X) &= g(H(X,Y) - H(Y,X), Z) \\ &+ g(H(Z,X),Y) - H(X,Z),Y) + g(H(Z,Y) - H(Y,Z), X) \\ &= 2(g(H(X,Y),Z) + g(\bar{K}_X Z, Y)). \end{aligned}$$

Substituting (1.10) in the last equation implies

$$\begin{aligned} g(H(X,Y),Z) = \frac{1}{2}\{&g(\eta(Y)\varphi X - \eta(X)\varphi Y, Z) + g(\eta(X)\varphi Z - \eta(Z)\varphi X, Y) \\ &+ g(\eta(Y)\varphi Z - \eta(Z)\varphi Y, X)\} - g(\bar{K}_X Z, Y). \end{aligned}$$

Thus we get

$$H(X,Y) = -\eta(X)\varphi Y - \bar{K}_X Y.$$

By taking the Equations (1.3) and (1.11) we get

$$\tilde{\nabla}_X Y = \bar{\nabla}^*_X Y - \eta(X)\varphi(Y) + K_X Y.$$

□

Theorem 2.3. *Let $(\bar{M}, \bar{\nabla}, g)$ be a cosymplectic statistical manifold admitting a quarter symmetric metric connection $\tilde{\nabla}$. Then $\tilde{\nabla}\varphi = 0$.*

Proof. For all $X, Y \in \mathcal{T}(\bar{M})$ by using Theorem 1.2 we have

$$\begin{aligned} (\tilde{\nabla}_X \varphi)Y = \tilde{\nabla}_X \varphi Y - \varphi \tilde{\nabla}_X Y &= \bar{\nabla}_X \varphi Y - \eta(X)\varphi^2 Y - K_X \varphi Y \\ &- \varphi \bar{\nabla}^*_X Y + \eta(X)\varphi^2 Y - \varphi K_X Y = 0. \end{aligned} \tag{2.2}$$

□

Lemma 2.4. *Let $(\bar{M}, \bar{\nabla}, g)$ be a cosymplectic statistical manifold admitting a quarter symmetric linear connection $\tilde{\nabla}$. Then the curvature tensor $\tilde{R}$ associated with $\tilde{\nabla}$ satisfies the following conditions for all $X, Y, Z, W \in \mathcal{T}(\bar{M})$*

1) $\tilde{R}(X,Y)Z = -\tilde{R}(Y,X)Z$,
2) $g(\tilde{R}(X,Y)Z, W) = -g(\tilde{R}(Y,X)W, Z)$.

We consider ∇' and h' the induced connection and second fundamental form on submanifold M with respect to the quarter symmetric metric connection $\tilde{\nabla}$, respectively. So the Gauss formula with respect to quarter symmetric metric connection $\tilde{\nabla}$ is the form $\tilde{\nabla}_X Y = \nabla'_X Y + h'(X,Y)$.

Theorem 2.5. *Let M be a φ-invariant submanifold of a cosymplectic statistical manifold $\bar{M}$ admitting a quarter symmetric metric connection $\tilde{\nabla}$ and $\xi \in \mathcal{T}(M)$. Then we have*

$$\begin{aligned}\nabla'_X Y =& \nabla_X Y - \eta(X)\varphi Y - K_X Y, \quad \forall X, Y \in \mathcal{T}(M),\\ h'(X,Y) =& \frac{1}{2}(h(X,Y) + h^*(X,Y)),\end{aligned}$$

where $K_X Y = \frac{1}{2}(\nabla - \nabla^*)$.

Proof. Applying (1.10) and Gauss formula in (1.7) we get

$$\begin{aligned}\tilde{\nabla}_X Y &= \bar{\nabla}_X Y - \eta(X)\varphi Y - \bar{K}_X Y = \nabla_X Y + h(X,Y) - \eta(X)\varphi Y\\ &\quad - \frac{1}{2}(\nabla_X Y + h(X,Y) - \nabla^*_X Y - h^*(X,Y))\\ &= \nabla_X Y - \eta(X)\varphi Y - K_X Y + \frac{1}{2}(h(X,Y) + h^*(X,Y)). \qquad (2.3)\end{aligned}$$

By separating the tangential and normal parts we get the result. □

Remark 2.6. *By similar proof of Theorem 2.5 we can show*

$$\nabla'_X Y = \nabla^*_X Y - \eta(X)\varphi Y + K_X Y, \quad \forall X, Y \in \mathcal{T}(M).$$

Corollary 2.7. *Let M be a φ-invariant submanifold of cosymplectic statistical manifold $\bar{M}$ such that $\bar{M}$ admits a quarter symmetric metric connection $\tilde{\nabla}$ and $\xi \in \mathcal{T}(M)$. Then the induced connection ∇' of quarter symmetric metric connection $\tilde{\nabla}$ is also quarter symmetric metric connection and $(\nabla'_X g)(Y,Z) = (\tilde{\nabla}_X g)(Y,Z)$.*

Corollary 2.8. *Let M be a φ-invariant submanifold of cosymplectic statistical manifold $\bar{M}$ such that $\bar{M}$ admits a quarter symmetric metric connection $\tilde{\nabla}$ and $\xi \in \mathcal{T}(M)$. Then the second fundamental form with respect to quarter symmetric metric connection $\tilde{\nabla}$ coincides with the second fundamental form of Levi-Civita connection.*

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GEODESICS ON THE LIGHTLIKE CONE

ALIREZA SEDAGHATDOOST AND NEMAT ABAZARI

Abstract. In this paper, we calculate radical distribution Rad TQ^n and lightlike transversal vector bundle of $\mathrm{ltr}(TQ^n)$ and by means of these, we prove that the geodesics of the lightlike cone Q^{n+1} are same as the geodesics of the Euclidean cone.

1. Introduction

Lightlike submanifolds have been one of the recently subjects in differential geometry and modern physics. In general relativity, lightlike submanifolds usually appear to be some smooth parts of the achronal boundaries. for example, event horizon of Kruskal and Kerr black holes and the compact Cauchy horizons in Taub-NUT spacetime.[3, 5, 9].

In a Riemannian or semi-Riemannian manifold the geodesics are fundamental tools for appreciation of the properties of the manifold. To find geodesics in a manifold plays a very important role in special and general relativity. The geodesics in space-time are classified as spacelike, lightlike or timelike. The physical significance of these types of geodesics that is the path of a light ray in space-time is described by a lightlike geodesic.

2. Perliminaries

Let E^n be n-dimensional Euclidean space. For two vectors $v = (v^1, \dots, v^n)$ and $w = (w^1, \dots, w^n)$ and an integer $0 \leq q \leq n$ the following bilinear form is defined.

$$\langle v, w\rangle_q := \sum_{i=1}^{n-q} v^i w^i - \sum_{i=n-q+1}^{n} v^i w^i, \tag{2.1}$$

that is a semi-Riemannian manifold. The resulting semi-Riemannian space is called Minkowski n-space; if $n = 4$ it is the simplest example of a relativistic space-time[9]. A vector $v \neq 0$ in E^n_q is called spacelike, timelike or lightlike if $\langle v, v\rangle_q > 0$, $\langle v, v\rangle_q < 0$ or $\langle v, v\rangle_q = 0$ respectively, and $v = 0$ is spacelike. The set of all lightlike vectors in E^{n+1}_q is called lightlike cone and denote by Q^n.

Let $(\bar{M}, \bar{g})$ be an n+1-dimensional semi-Riemannian manifold and (M, g) be a hypersurface of $\bar{M}$ with degenerate metric g induced of $\bar{g}$ by an immersion $i : M \to \bar{M}$. Since g is degenerate on M, thus there is a vector field $\xi \neq 0$ on M such that

$$g(\xi, X) = 0, \quad \forall X \in \Gamma(TM)$$

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For any $x \in M$, the radical space of T_xM is a subspace of T_xM, defined by

$$\mathrm{Rad}T_xM := \{\xi \in T_xM \ : \ g_x(\xi, X) = 0, \ \forall X \in T_xM\} \tag{2.2}$$

and M is called a lightlike hypersurface of $\bar{M}$[9].

In a lightlike hypersurface, standard definition of second fundamental form and Gauss-Weingarten formulas do not work; thus in 1991 Bejancu-Duggal in [2] introduced a new technique as follow.

Let $S(TM)$ be complementary of $\mathrm{Rad}TM$ in TM, thus $S(TM)$ is a non-degenerate distribution of M which is called screen distribution on M, since M is paracompact thus there always exists a screen distribution $S(TM)$. For this reason we can decompose $T(\bar{M})$ to orthogonal direct sum of two sub-bundle $S(TM)$ and $S(TM)^{\perp}$ (For more details, see [2, 5, 4]).

Theorem 2.1 ([5])**.** *Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $(\bar{M}, \bar{g})$.Then there exists a unique bundle $ltr(TM)$ of rank 1 over M, such that for any non-zero section ξ of $TM^{\perp}$ on a coordinate neighborhood $U \subset M$, there exists a unique section N of ltr(TM) on U satisfying:*

$$\bar{g}(N, \xi) = 1, \ \bar{g}(N, N) = \bar{g}(N, W) = 0, \ \forall W \in \Gamma(S(TM)_{|_U}). \tag{2.3}$$

Now we can decompose $T(\bar{M}_{|_M})$ as follow.

$$T\bar{M}_{|_M} = S(TM) \perp (\mathrm{Rad}TM \oplus ltr(TM)) = TM \oplus ltr(TM), \tag{2.4}$$

that we call $ltr(TM)$ the lightlike transversal vector bundle of M with respect to $S(TM)$.

By using the second form of decomposition in (2.4) we obtain Gauss and Weingarten formulas as follow

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.5}$$

$$\bar{\nabla}_X V = -A_V X + \nabla^t_X V, \tag{2.6}$$

for all $X, Y \in \Gamma(TM)$ and $V \in \Gamma(ltr(TM))$, where $\nabla_X Y$ and $A_V X$ belong to $\Gamma(TM)$ while $h(X, Y)$ and $\nabla^t_X V$ belong to $\Gamma(ltr(TM))$ [5].
Since rank of $\Gamma(ltr(TM))$ is 1 thus if we set

$$B(X, Y) := \bar{g}(\bar{\nabla}_X Y, \xi), \tag{2.7}$$

$$\tau(X) = \bar{g}(\nabla^t_X N, \xi), \tag{2.8}$$

then by theorem 2.1 in the relations (2.5), (2.6) we conclude that

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \tag{2.9}$$

$$\bar{\nabla}_X V = -A_V X + \tau(X)N. \tag{2.10}$$

One of the example in the lightlike hypersurface is n-dimensional lightlike cone of Minkowski space E^{n+1}_q. Let $(E^{n+1}_q, \langle \ , \ \rangle_q)$ be $(n+1)$-dimensional Pseudo-Euclidean space that its metric be defined in (2.1). $(E^{n+1}_q, \langle \ , \ \rangle_q)$ is called Minkowski space of index q [9].
The upper lightlike hypersurface Q^n_q of E^{n+1}_q be viewed by

$$i : E^n \to E^{n+1}_q,$$

$$i(x_1, \ldots, x_n) := (x_1, \ldots, x_n, b), \tag{2.11}$$

where $b = \sqrt{\sum_{j=1}^{n-q+1} x_j^2 - \sum_{j=n-q+2}^{n} x_j^2}$. The case Q^n_1 denoted by Q^n.

3. ABOUT THE CURVES TYPE OF LIGHTLIKE CONE OF INDEX 1

In Euclidean space a regular curve is a curve that its velocity vector is nonzero. In the Minkowski space E_1^3, any timelike (lightlike) curve is regular. Also if a curve $x : I \to E_1^3$ is regular in s_0 then by continuity, x is regular in a neighborhood of s_0[7]. Similarly to this, we can prove the following.

Proposition 3.1. *Any timelike (lightlike) curve $x : I \to E_1^{n+1}$ (with arbitrary parameter) is regular.*

Proof. Assume that the curve is timelike. We write

$$x(t) = (x_1(t), \dots, x_n(t), x_{n+1}(t)),$$

where $x_i(t)$ are differentiable functions on I. In this case we have

$$\langle \dot{x}(t), \dot{x}(t)\rangle = \dot{x}_1^2(t) + \cdots + \dot{x}_n^2(t) - \dot{x}_{n+1}^2(t) < 0, \tag{3.1}$$

in particular $\dot{x}_{n+1}(t) \neq 0$, that is, x is regular.
If the curve is lightlike, we have $\dot{x}_{n+1}(t) \neq 0$ again since, on the contrary, $\dot{x}_i(t) = 0$ and $\dot{x}(t) = 0$. But this means that the curve is spacelike. □

Lemma 3.2. *Let $x : I \to Q^n \subset E_1^{n+1}$ be a curve. Then x is lightlike if and only if x is a straight line.*

Proof. Let $\langle x, x\rangle = 0$ and $\langle \dot{x}, \dot{x}\rangle = 0$ that is

$$x_{n+1}^2 = x_1^2 + \cdots + x_n^2, \tag{3.2}$$

$$\dot{x}_{n+1}^2 = \dot{x}_1^2 + \cdots + \dot{x}_n^2. \tag{3.3}$$

By differentiation of (3.2) yields that

$$\begin{gathered} x_{n+1}\dot{x}_{n+1} = x_1\dot{x}_1 + \cdots + x_n\dot{x}_n, \\ (x_{n+1})^2(\dot{x}_{n+1})^2 = (x_1\dot{x}_1 + \cdots + x_n\dot{x}_n)^2. \end{gathered} \tag{3.4}$$

By replace of (3.2) and (3.3) in (3.4) and some calculations we conclude that:

$$\sum_{i,j=1}^{n} (x_i\dot{x}_j - x_j\dot{x}_i)^2 = 0,$$

$$x_i\dot{x}_j - x_j\dot{x}_i = 0,$$

$$\frac{\dot{x}_i}{x_i} = \frac{\dot{x}_1}{x_1}\ ,\ i = 1, \dots, n,$$

$$x_i(s) = A_i x_1(s); \quad i = 1, \dots, n, \quad x_{n+1}(s) = \pm\sqrt{1 + A_1^2 + \cdots + A_n^2}\ x(s),$$

that A_i is some real constant. Thus $x(s) = \overrightarrow{A}x_1(s)$ that is a straight line with real differential function $x_1(s)$ and constant lightlike velocity vector $\overrightarrow{A}$. □

Lemma 3.3. *Let $x : I \to E_1^{n+1}$ be a timelike curve. Then x is not lying in the Q^n.*

Proof. If x be in the Q^n, then

$$x_{n+1}^2 = x_1^2 + \cdots + x_n^2,$$

therefor

$$\sum_{i=1}^{n} x_i x_i' = x_{n+1}x_{x+1}'. \tag{3.5}$$

and since x is timelike thus:

$$(x_1^2 + \cdots + x_n^2)({x'}_1^2 + \cdots + {x'}_n^2) - x_{n+1}^2 {x'}_{x+1}^2 = -x_{n+1}^2. \tag{3.6}$$

If we replace (3.5) in (3.6) then

$$\sum_{i,j=1}^{n} (x_i {x'}_j - x_j {x'}_i)^2 = -x_{n+1}^2,$$

thus $x_{n+1} = 0$ and by (3.5) yield that $x_i(s) = 0;\ i = 1, \ldots, n$. So $x(s) = 0$ that is a contradiction. □

These two lemmas yield the following theorem.

Theorem 3.4. *Let $x : I \to Q^n \subset E_1^{n+1}$ be a regular curve. then x is non-straight line if and only if x is a spacelike curve.*

Proof. Let x be a non-straight line curve in Q^n, then by lemma 3.2 this curve is not a lightlike curve and by lemma 3.3 this curve is not a timelike curve, thus it is a spacelike curve.
Conversely; if the curve is a spacelike and straight line then $x(s) = \overrightarrow{A}\tilde{x}(s)$ that $\tilde{x}(s)$ is a real differential function and $\overrightarrow{A}$ is a lightlike vector as x is lightlike, that is a contradiction. □

4. Rad TQ_q^n and $ltr(TQ_q^n)$

Since induced metric on lightlike cone Q^n of Minkowski space E_1^{n+1} is degenerate thus Rad TQ^n is nontrivial and attention to theorem 2.1 there exists an unique transversal vector bundle of rank 1 over M. We calculate Rad (TQ^n) and $ltr(TQ^n)$ in the following theorem.

Theorem 4.1. *Let Q^n be n-dimensional lightlike cone of Minkowski space E_q^{n+1} defined in (2.11). Then radical distribution of Rad T_xQ^n spaned by $\xi = \sum_{j=1}^{n} x_j\partial_j + b\partial_{n+1}$, corresponding to ξ, unique section N of ltr(TQ^n) in theorem 2.1 is*

$$N = \frac{1}{2b^2}\left(\sum_{j=1}^{n} x_j\partial_j - b\partial_{n+1}\right), \tag{4.1}$$

and screen distribution $S(TQ^n)$ is

$$S(T_xQ^n) = Span\{W_j \ :\ 1 \le j \le n-1\} \tag{4.2}$$

where $W_j = x_{j+1}\partial_1 - x_1\partial_{j+1}$ for $1 \le j \le n-q$ and $W_j = x_{j+1}\partial_1 + x_1\partial_{j+1}$ for $n-q+1 \le j \le n-1$.

Proof. Considering (2.11), the tangent space of Q_q^n at $i(x)$ spanned by following vectors

$$\frac{\partial i}{\partial x_j} = \partial_j + \frac{\partial b}{\partial x_j}\partial_{n+1} = \partial_j + \frac{x_j}{b}\partial_{n+1}, \quad (1 \le j \le n-q+1),$$
$$\frac{\partial i}{\partial x_j} = \partial_j - \frac{\partial b}{\partial x_j}\partial_{n+1} = \partial_j - \frac{x_j}{b}\partial_{n+1}, \quad (n-q+2 \le j \le n),$$

Set $V_j = b\partial_j + x_j\partial_{n+1}$ for $1 \le j \le n-q+1$ and $V_j = b\partial_j - x_j\partial_{n+1}$ for $n-q+2 \le j \le n$, then

$$T_xQ^n = span\{V_j \ :\ 1 \le j \le n\}.$$

Let $\xi=\sum_{j=1}^{n+1}\xi^j\partial_j$ be a vector of Rad TQ_q^n, then ξ satisfy in the equations

$$\langle \xi, V_j\rangle_q = 0, \quad 1\le j\le n,$$
$$\langle\sum_{k=1}^{n+1}\xi^k\partial_k,\ b\partial_j + x_j\partial_{n+1}\rangle_q = 0, \quad (1\le j\le n-q),$$
$$\langle\sum_{k=1}^{n+1}\xi^k\partial_k,\ b\partial_j - x_j\partial_{n+1}\rangle_q = 0, \quad (n-q+1\le j< n-1),$$
$$\xi^j = \frac{\xi^{n+1}x_j}{b}, \quad 1\le j\le n.$$

If we choose $\xi^{n+1}=b$ then $\xi^j=x_j, 1\le j\le n$ and Rad $T_xQ_q^n$ spanned by $\xi=\sum_{j=1}^{n}x_j\partial_j+b\partial_{n+1}$.
Set

$$W_j := \frac{1}{b}(x_{j+1}V_1 - x_1V_{j+1}) = x_{j+1}\partial_1 - x_1\partial_{j+1}, \quad 1\le j\le n-q, \tag{4.3}$$
$$W_j := \frac{1}{b}(x_{j+1}V_1 + x_1V_{j+1}) = x_{j+1}\partial_1 + x_1\partial_{j+1}, \quad n-q+1\le j< n-1, \tag{4.4}$$

and since $\{W_1,\dots,W_{n-1}\}$ is linearly independent thus screen distribution of Q_q^n is

$$S(TQ^n) = span\,\{W_j\ :\ 1\le j\le n-1\}\,. \tag{4.5}$$

Let $N=\sum_{i=1}^{n+1}\eta^i\partial_i\in ltr(TQ_q^n)$, be unique section of $ltr(TQ_q^n)$ in theorem 2.1. Thus

$$\sum_{i=1}^{n-q+1}(\eta^i)^2 - \sum_{i=n-q+2}^{n}(\eta^i)^2 = 0, \tag{4.6}$$
$$\sum_{i=1}^{n-q+1}x_i\eta^i - \sum_{i=n-q+2}^{n}x_i\eta^i - b\eta^{n+1} = 1, \tag{4.7}$$
$$\eta^1x_{i+1} - \eta^{i+1}x_1 = 0. \tag{4.8}$$

From (4.6) we deduce that

$$\eta^{n+1} = \varepsilon\sqrt{\sum_{i=1}^{n}(\eta^i)^2 - \sum_{i=n-q+2}^{n}(\eta^i)^2}, \quad (\varepsilon=+1 \text{ or } -1). \tag{4.9}$$

By relation (4.8) we have

$$\eta^{j+1} = \frac{x_{j+1}}{x_1}\eta^1, \tag{4.10}$$

that if it replaces in (4.9) then

$$\eta^{n+1} = \varepsilon b\frac{\eta^1}{x_1}. \tag{4.11}$$

Now if we replace (4.11) in (4.7) then

$$x_1\eta^1 + \sum_{i=1}^{n-q}\frac{(x_{i+1})^2}{x_1}\eta^1 - \sum_{i=n-q+1}^{n-1}\frac{(x_{i+1})^2}{x_1}\eta^1 - \varepsilon b^2\frac{\eta^1}{x_1} = 1,$$
$$(b^2-\varepsilon b^2)\eta^1 = x_1,$$

thus $\varepsilon = -1$, $\eta^1 = \dfrac{x_1}{2b^2}$ seduce that

$$\eta^{n+1} = -\frac{1}{2b}\ ,\ \eta^i = \frac{x_i}{2b^2}\ ,\ 1 \leq i \leq n, \tag{4.12}$$

$$N = \frac{1}{2b^2}\left(\sum_{i=1}^{n} x_i \partial_i - b\partial_{n+1}\right). \tag{4.13}$$

□

5. Geodesics of Lightlike Cone

In this section, we will show that the geodesics of lightlike cone with respect to the both Euclidean and Minkowski metrics are same, thus Minkowski metric $\langle\ ,\ \rangle_q$ is *geodesically equivalent* [8] to Euclidean metric $\langle\ ,\ \rangle$.

Lemma 5.1. *Let EQ^n be the cone with Euclidean metric, then lightlike transversal vector in* (4.1) *is normal vector of hypersurface EQ^n.*

Proof. In theorem 2.1 we prove that the tangent space of Q^n_q be spanned by the vectors $V_j = b\partial_j + x_j\partial_{n+1}$ for $1 \leq j \leq n-q+1$ and $V_j = b\partial_j - x_j\partial_{n+1}$ for $n-q+2 \leq j \leq n$, furthermore

$$\langle N, V_j\rangle = \langle\frac{1}{2b^2}\left(\sum_{j=1}^{n} x_j\partial_j - b\partial_{n+1}\right), x_j\partial_0 + b\partial_j\rangle = 0,$$

that $\langle\ ,\ \rangle$ is Euclidean inner product of E^{n+1}. □

The lemma 5.1 is valid for any lightlike hypersurface of Minkowski space E^{n+1}_q (see [1], proposition 3.2), but we proved for special lightlike hypersurface Q^n_q in different way by direct calculations.

Theorem 5.2. *Let Q^n_q be lightlike cone of Minkowski space E^{n+1}_q. Then with respect to the induced metric of E^{n+1}_q, the geodesics of lightlike cone Q^n_q are same to the geodesics of this cone with respect to the induced metric of Euclidean space E^{n+1}.*

Proof. Let $\alpha : I \to Q^n$ be a curve in n-dimensional lightlike cone. From (2.9), we have

$$\alpha'' = \nabla_{\alpha'}\alpha' + B(\alpha', \alpha')N. \tag{5.1}$$

If α be a geodesics of Q^n then $\alpha'' = B(\alpha', \alpha')N$, and since in the Euclidean hypersurfaces the acceleration of a geodesic is in normal space of hypersurface, thus by using of lemma 5.1, we have $\alpha : I \to Q^n$ is also a geodesic of EQ^n.

□

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ON SYMMETRIES OF THE PSEUDO-RIEMANNIAN MANIFOLD $\mathbb{S}^2 \times \mathbb{R}$

PARVANE ATASHPEYKAR AND ALI HAJI-BADALI

Abstract. In this article, we studied the symmetries of the pseudo-Riemannian manifold $\mathbb{S}^2 \times \mathbb{R}$. Specially, we perused the existence of Ricci and matter collineations in this space.

Key words and phrases: Pseudo-Riemannian metric; Killing and affine vector field; Ricci and matter collineation.

1. Introduction

The study of symmetries in general relativity has long been considered due to they are interesting both from the mathematical and the physical point of view (see for example [6]). The symmetry is a one-parameter group of diffeomorphisms of the pseudo-Riemannian manifold (M, g), which leaves a special mathematical or physical quantity invariant. This statement is equivalent to the Lie derivative of the geometry quantity under the vector field X vanishes, i.e., $\mathcal{L}_X \mathcal{S} = 0$. If $\mathcal{S}$ has geometrical or physical significance, then those special vector fields under which $\mathcal{S}$ is invariant will also be of significance. Isometries, homotheties, and conformal motions are well-known examples of symmetries. Recently, other types of symmetries including curvature collineations ($\mathcal{S} = \mathcal{R}$ being the curvature tensor), Ricci collineations ($\mathcal{S} = \varrho$ being the Ricci tensor), and etc., have been studied. Some examples may be found in [1, 2].

On the pseudo-Riemannian manifold (M, g) a matter collineation is a vector field X, which preserves the energy-momentum tensor $\mathcal{S} = \varrho - \dfrac{\tau}{2} g$, where τ shows the scalar curvature. Since the Ricci tensor is constructed from the connection of the metric tensor, Ricci collineations have geometrical importance. However, matter collineations are more related to a physical viewpoint [3, 4]. These physical and geometric concepts give a single meaning in a particular case, for example, when the meter tensor has a zero scalar curvature.

In this article, we study symmetries of the pseudo-Riemannian manifold $\mathbb{S}^2 \times \mathbb{R}$. We present a complete classification of its Ricci and matter collineations. Clearly, any Killing vector field (respectively, any affine vector field and curvature collineation) is an affine vector field (respectively, any curvature and Ricci collineation) but the inverse is always not true. Also, a homothetic vector field (i.e., a vector field that holds in relation $\mathcal{L}_X g = \eta g$, where η is a real number) is a Ricci collineation. Thus, we examine the existence of proper Ricci and curvature collineations, which are not Killing and homothetic. $Maple16^{©}$ is used to check all computations.

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2. The pseudo-Riemannian manifold $\mathbb{S}^2 \times \mathbb{R}$

We consider the local model for $\mathbb{S}^2 \times \mathbb{R}$ given by $\mathbb{R}^3$ endowed with the pseudo-Riemannian metric

$$g = \frac{1}{(1+x^2+y^2)^2}(dx^2+dy^2) - dz^2. \tag{2.1}$$

Then, the non-zero components of the Levi-Civita connection ∇ of the pseudo-Riemannian manifold $\mathbb{S}^2 \times \mathbb{R}$ are given by

$$\begin{aligned}
\nabla_{\partial_x}\partial_x &= -\frac{2x}{(1+x^2+y^2)}\partial_x + \frac{2y}{(1+x^2+y^2)}\partial_y,\\
\nabla_{\partial_x}\partial_y &= -\frac{2y}{(1+x^2+y^2)}\partial_x - \frac{2x}{(1+x^2+y^2)}\partial_y,\\
\nabla_{\partial_y}\partial_y &= \frac{2x}{(1+x^2+y^2)}\partial_x - \frac{2y}{(1+x^2+y^2)}\partial_y,
\end{aligned}$$

and the non-zero component of the curvature tensor $\mathcal{R}$ is

$$\mathcal{R}(\partial_x,\partial_y)\partial_y = \frac{4}{(1+x^2+y^2)^2}\partial_x. \tag{2.2}$$

Also, the non-zero components of the Ricci tensor are $\rho_{11} = \rho_{22} = \frac{4}{(1+x^2+y^2)^2}$.

3. Symmetries of $\mathbb{S}^2 \times \mathbb{R}$

The classification of Killing and affine vector fields on the pseudo-Riemannian manifold $(\mathbb{S}^2 \times \mathbb{R}, g)$ is as in the following theorem.

Theorem 3.1. *Assume $X = X^1\partial_x + X^2\partial_y + X^3\partial_z$ be an arbitrary vector field and ψ be a smooth function on the pseudo-Riemannian manifold $(\mathbb{S}^2 \times \mathbb{R}, g)$. Then*

(i) *X is a Killing vector field if and only if*

$$\begin{aligned}
X^1 &= \frac{1}{2}c_1(1+x^2-y^2) - (2c_2x+c_3)y,\\
X^2 &= -c_2(1-x^2+y^2) + (c_1y+c_3)x,\\
X^3 &= c_4.
\end{aligned}$$

(ii) *X is an affine, non-Killing vector field if and only if*

$$\begin{aligned}
X^1 &= -c_1(1+x^2-y^2) + (c_2x+c_3)y,\\
X^2 &= \frac{1}{2}c_2(1-x^2+y^2) - (2c_1y+c_3)x,\\
X^3 &= c_4z+c_5.
\end{aligned}$$

In the above expressions, c_i is an arbitrary real number, for any indices i.

Proof. A straightforward computation displays that the Lie derivative of g is given by

$$\begin{aligned}\mathcal{L}_X g =&\frac{8}{(1+x^2+y^2)^3}((1+x^2+y^2)\partial_x X^1 - 2xX^1 - 2yX^2)dxdx\\&+\frac{8}{(1+x^2+y^2)^2}(\partial_x X^2+\partial_y X^1)dxdy\\&-\frac{2}{(1+x^2+y^2)^2}((1+2x^2+2y^2+2x^2y^2+x^4+y^4)\partial_x X^3 - 4\partial_z X^1)dxdz\\&+\frac{8}{(1+x^2+y^2)^3}((1+x^2+y^2)\partial_y X^2 - 2xX^1 - 2yX^2)dydy\\&-\frac{2}{(1+x^2+y^2)^2}((1+2x^2+2y^2+2x^2y^2+x^4+y^4)\partial_y X^3 - 4\partial_z X^2)dydz\\&-2\partial_z X^3 dzdz.\end{aligned}$$

To obtain Killing vector fields, we put all the coefficients of the $\mathcal{L}_X g$ equal to zero and solve the corresponding system of partial differential equations. The solutions of this system give case (i).

Affine vector fields are determined by solving the system of PDEs, obtained from the vanishing of the coefficients of the $\mathcal{L}_X \nabla$. This proves the case (ii).

□

Next, we will focus on symmetries of $(\mathbb{S}^2\times\mathbb{R}, g)$ relative to curvature. The results are reported in the following theorem.

Theorem 3.2. *Assume $X = X^1\partial_x + X^2\partial_y + X^3\partial_z$ be an arbitrary vector field on the pseudo-Riemannian manifold $(\mathbb{S}^2\times\mathbb{R}, g)$. Then*

(i) *X is a Ricci collineation if and only if X^3 is arbitrary and*

$$X^1 = \frac{1}{2}c_1(1+x^2-y^2) - (2c_2x+c_3)y, \quad X^2 = -c_1(1-x^2+y^2) + (c_1y+c_3)x.$$

(ii) *X is a curvature collineation if and only if*

$$\begin{aligned}X^1 &= \frac{1}{2}c_1(1+x^2-y^2) + (2c_2x+c_3)y,\\X^2 &= -c_1(1-x^2+y^2) + (c_1y+c_3)x,\\X^3 &= f(z),\end{aligned}$$

where $f(z)$ is an arbitrary smooth function on $\mathbb{S}^2\times\mathbb{R}$.

Proof. The Lie derivative of the Ricci tensor in the direction X is determined by

$$\begin{aligned}(\mathcal{L}_X\varrho) =&\frac{8}{(1+x^2+y^2)^3}((1+x^2+y^2)\partial_x X^1 - 2xX^1 - 2yX^2)dxdx\\&+\frac{8}{(1+x^2+y^2)^2}(\partial_x X^2+\partial_y X^1)dxdy+\frac{8}{(1+x^2+y^2)^2}(\partial_z X^1)dxdz\\&+\frac{8}{(1+x^2+y^2)^3}((1+x^2+y^2)\partial_y X^2 - 2xX^1 - 2yX^2)dydy\\&+\frac{8}{(1+x^2+y^2)^2}(\partial_z X^2)dydz.\end{aligned}$$

Now, we need to put the coefficients of the $\mathcal{L}_X \varrho$ equivalent to zero and solve the corresponding system of partial differential equations to obtain the Ricci collineations. The solutions to this system give case (i).

Next, we investigate curvature collineations, beginning from an arbitrary Ricci collineation and apply the extra condition $\mathcal{L}_X \mathcal{R} = 0$. Thus,

$$X = \frac{1}{2} c_1 (x^2 - y^2) \partial_x + (c_1 x + c_2) y \partial_y + X^3 \partial_z,$$

is also a curvature collineation if and only if

$$\frac{1}{(1+x^2+y^2)^2} \partial_x X^3 = \frac{1}{(1+x^2+y^2)^2} \partial_y X^3 = 0,$$

which gives the result case (ii). □

Now, we classify matter collineations on the pseudo-Riemannian manifold $(\mathbb{S}^2 \times \mathbb{R}, g)$.

Theorem 3.3. *Assume $X = X^1 \partial_x + X^2 \partial_y + X^3 \partial_z$ be an arbitrary smooth vector field on the pseudo-Riemannian manifold $(\mathbb{S}^2 \times \mathbb{R}, g)$. Then, X is a matter collineation if and only if X^1, X^2 are arbitrary and $X^3 = c$, where c is a real constant.*

Proof. A straightforward computation displays that only the non-zero component of the tensor field $\mathcal{S}$ is $\mathcal{S}(\partial_z, \partial_z) = -1$. Now, we compute the Lie derivative of the tensor field $\mathcal{S}$. We have

$$\mathcal{L}_X \mathcal{S} = -2\partial_x X^3 dxdy - 2\partial_y X^3 dydz - 2\partial_z X^3 dzdz.$$

Requiring that $\mathcal{L}_X \mathcal{S} = 0$ we attain the system of partial differential equations, which solutions specify the matter collineations of $(\mathbb{S}^2 \times \mathbb{R}, g)$. Thus, X is a matter collineation if and only if X^1, X^2 are arbitrary and X^3 is a real constant and this completes the proof. □

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TAKAHASHI'S THEOREM ON HYPERSURFACES OF MINKOWSKI SPACES

FIROOZ PASHAIE AND LEILA SHAHBAZ

Abstract. In this paper, we classify timelike hypersurfaces in Lorentz-Minkowski space, $x : M^n \to \mathbb{L}^{n+1}$, satisfying the condition $L_k x = Ax + b$, where L_k is the kth extension of Laplace operator (i.e. Δ), A is a constant matrix and b is a constant vector. The condition $L_k x = Ax + b$ is a new version of a well-known equation $\Delta x = dx$ for a real number d. As an extension of Takahashi's theorem we show that such a hypersurface has to be k-minimal or an open piece of $\mathbb{S}^n_1(c)$, $\mathbb{S}^m_1(c) \times \mathbb{R}^{n-m}$ or $\mathbb{S}^m(c) \times \mathbb{L}^{n-m}$ for some $c > 0$ and $1 < m < n$.

Key words and phrases: Timelike hypersurface; Higher order mean curvature; Lorentz-Minkowski space.

1. Introduction

In 1966, Takahashi [5], determined the n-dimensional submanifolds isometrically immersed into the Euclidean space $\mathbb{R}^{n+m}$ whose position vector field is an eigenvector of the Laplace operator Δ with the same eigenvalue. In particular, by Takahashi Theorem, an immersed hypersurface $\psi : M^n \to \mathbb{R}^{n+1}$ satisfies the condition $\Delta\psi = \lambda\psi$ for a real λ if and only if either $\lambda = 0$ and M is minimal in $\mathbb{R}^{n+1}$ or $\lambda > 0$ and M is an open subset of the hypersphere of radius $\sqrt{\dfrac{n}{\lambda}}$ centered at the origin of $\mathbb{R}^{n+1}$. Many people generalized this result in different directions. In 1990, Garay [4], studied hypersurfaces $\psi : M^n \to \mathbb{R}^{n+1}$ satisfying the extended condition $\Delta\psi = D\psi$ where D is a diagonal matrix, and he proved that such hypersurfaces are minimal hypersurface and open pieces of either round hyperspheres or generalized right spherical cylinders. Dillen et al.[3] studied surfaces in $\mathbb{R}^3$ whose position vector satisfies an extended condition $\Delta\psi = A\psi + b$ where $A \in \mathbb{R}^{3\times 3}$ is a constant matrix, $b \in \mathbb{R}^3$ is a constant vector. He proved that such surfaces are minimal surfaces and open pieces of spheres and circular cylinders in $\mathbb{R}^3$. Alias et al.[2] considered hypersurfaces in space forms whose position vector field satisfy the general condition $L_k\psi = A\psi + b$, for a fixed integer $0 \leq k \leq n-1$, a matrix $A \in \mathbb{R}^{(n+1)\times(n+1)}$ and a vector $b \in \mathbb{R}^{n+1}$. He and N. G*ü*rb*ü*z [1] classified hypersurfaces in the Euclidean space $\mathbb{R}^{n+1}$ whose position vector field satisfy the general condition $L_k\psi = A\psi + b$.

Based on this background, we consider the timelike hypersurfaces in the Lorentz-Minkowski space whose position vector field satisfies $L_k\psi = A\psi + b$ for a matrix A and a vector b. It is well known that for a Lorentzian hypersurface M, its shape operator S is not necessarily diagonalizable. In spite of this difficulty, we classify such hypersurfaces.

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2. Preliminaries

Now we remember some notations and give the main definitions. The m-dimensional pseudo-Euclidean space $\mathbb{R}_q^m$ of index q, stands for the vector space $\mathbb{R}^m$ with the scalar product $< x, y >:= -\Sigma_{i=1}^q x_i y_i + \Sigma_{j>q} x_j y_j$, where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$.

For $r \neq 0$, the non-flat space form of curvature r is

$$\mathbb{M}_q^{n+1}(r) = \begin{cases} \mathbb{S}_q^{n+1}(\frac{1}{\sqrt{r}}) = \{y \in \mathbb{R}_q^{n+2} | < y, y >= \dfrac{1}{r}\}, \ (r > 0), \\ H_q^{n+1}(\frac{-1}{\sqrt{-r}}) = \{y \in \mathbb{R}_{q+1}^{n+2} | < y, y >= \dfrac{1}{r}\}, \ (r < 0). \end{cases}$$

For the hypersurface in the Euclidean space the shape operator S, associated to a chosen (local) normal vector field $\mathbf{n}$ on M, is diagonalizable, but it is not necessarily true For the timelike hypersurface in the Minkowski space-time. In the special case that S is diagonalizable with the eigenvalue functions $\kappa_1, \ldots, \kappa_n$ on M, using $s_j := \sum_{1 \leq i_1 < \cdots < i_j \leq n} \kappa_{i_1} \ldots \kappa_{i_j}$, the jth mean curvature of M is defined by $\binom{n}{j} H_j = (-\epsilon)^j s_j$, where $\epsilon := - < \mathbf{n}, \mathbf{n} >$.

Definition 2.1. The Newton transformations $P_j : \chi(M) \to \chi(M)$, is defined by

$$P_0 = I, P_j = (-\epsilon)^j s_j I + \epsilon S \circ P_{j-1} (j = 1, \ldots, n), \ (I = Id_{\chi(M)}).$$

When S is diagonalizable, the Newton transformation P_j is self-adjoint and diagonalizable on M, and commutes with S.

We generalize the notions of H_j and P_j to timelike hypersurfaces in the Minkowski space.

Proposition 2.2. Let $x : M^n \to \mathbb{L}^{n+1}$ (where $n \geq 2$) be a connected timelike hypersurface isometrically immersed into the Minkowski space and P_k be the kth Newton transformation. If at a point $p \in M$, $H_k(p) = 0$ and $H_{k+1}(p) \neq 0$, then P_{k-1} is definite at p.

Definition 2.3. The *linearized operator* of the $(k + 1)$th mean curvature of M, $L_k : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$ is defined by the formula $L_k(f) := tr(P_k \circ \nabla^2 f)$, where,

$$< \nabla^2 f(X), Y >=< \nabla_X grad(f), Y >,$$

for every $X, Y \in \chi(M)$.

Throughout the paper, we study on every Lorentzian hypersurface of $\mathbb{L}^{n+1}$, defined by an isometric immersion $x : M_1^n \to \mathbb{L}^{n+1}$. The symbols $\tilde{\nabla}$ and $\bar{\nabla}$ stand for the Levi-Civita connection on M_1^n and $\mathbb{L}^{n+1}$, respectively. For every tangent vector fields X and Y on M, the Gauss formula is given by

$$\bar{\nabla}_X Y = \tilde{\nabla}_X Y + < SX, Y > \mathbf{n},$$

for every $X, Y \in \chi(M)$. For each non-zero vector $X \in \mathbb{L}^{n+1}$, the real value $< X, X >$ may be a negative, zero or positive number and then, the vector X is said to be time-like, light-like or space-like, respectively.

Definition 2.4. For a n-dimensional Lorentzian vector space V_1^n, a basis $\mathcal{B} := \{e_1, \ldots, e_n\}$ is said to be *orthonormal* if it satisfies $< e_i, e_j >= \epsilon_i \delta_i^j$ for $i, j = 1, \ldots, n$, where $\epsilon_1 = -1$ and $\epsilon_i = 1$ for $i = 2, \ldots, n$. As usual, δ_i^j stands for the Kronecker delta. $\mathcal{B}$ is called *pseudo-orthonormal* if it satisfies

$$< e_1, e_1 >=< e_2, e_2 >= 0, \quad < e_1, e_2 >= -1,$$

and $< e_i, e_j >= \delta_i^j$, for $i = 1, \dots, n$ and $j = 3, \dots, n$.

As well-known, the shape operator A of the Lorentzian hypersurface M_1^n in $\mathbb{L}^{n+1}$, as a self-adjoint linear map on the tangent bundle of M_1^n, locally can be put into one of four possible canonical matrix forms, usually denoted by I, II, III and IV. Where, in cases I and IV, with respect to an orthonormal basis of the tangent space of M_1^n, the matrix representation of the induced metric on M_1^n is $G_1 = diag_n[-1, 1, \dots, 1]$ and the shape operator of M_1^n can be put into matrix forms $B_1 = diag[\lambda_1, \dots, \lambda_n]$ and $B_4 = diag[\left[\begin{smallmatrix} \kappa & \lambda \\ -\lambda & \kappa \end{smallmatrix}\right], \eta_1, \dots, \eta_{n-2}]$, (where $\lambda \neq 0$), respectively. For cases II and III, using a pseudo-orthonormal basis of the tangent space of M_1^n, the induced metric on which has matrix form $G_2 = diag_n[\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right], 1, \dots, 1]$ and the shape operator of M_1^n can be put into matrix forms $B_2 = diag_n[\left[\begin{smallmatrix} \kappa & 0 \\ 1 & \kappa \end{smallmatrix}\right], \lambda_1, \dots, \lambda_{n-2}]$ and $B_3 = diag_n[\left[\begin{smallmatrix} \kappa & 0 & 0 \\ 0 & \kappa & 1 \\ -1 & 0 & \kappa \end{smallmatrix}\right], \lambda_1, \dots, \lambda_{n-3}]$, respectively. In case IV, the matrix B_4 has two conjugate complex eigenvalues $\kappa \pm i\lambda$, but in other cases the eigenvalues of the shape operator are real numbers.

Remark 2.5. In two cases II and III, one can substitute the pseudo-orthonormal basis $\mathcal{B} := \{e_1, e_2, \dots, e_n\}$ by a new orthonormal basis $\tilde{\mathcal{B}} := \{\tilde{e_1}, \tilde{e_2}, e_3, \dots, e_n\}$ where $\tilde{e_1} := \dfrac{1}{2}(e_1 + e_2)$ and $\tilde{e_2} := \dfrac{1}{2}(e_1 - e_2)$. Therefore, we obtain new matrices $\tilde{B}_2$ and $\tilde{B}_3$ (instead of B_2 and B_3, respectively) as $\tilde{B}_2 = diag_n[\left[\begin{smallmatrix} \kappa + \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \kappa - \frac{1}{2} \end{smallmatrix}\right], \lambda_1, \dots, \lambda_{n-2}]$ and $\tilde{B}_3 = diag_n[\left[\begin{smallmatrix} \kappa & 0 & \frac{\sqrt{2}}{2} \\ 0 & \kappa & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & \kappa \end{smallmatrix}\right], \lambda_1, \dots, \lambda_{n-3}]$. After this changes, to unify the notations we denote the orthonormal basis by $\mathcal{B}$ in all cases.

3. Main results

Theorem 3.1 ([1])**.** *Let $x : M^n \to \mathbb{R}^{n+1}$ be an orientable hypersurface immersed into the Euclidean space. Then, x satisfies $L_k x = Ax + b$, for an integer $0 \leq k < n$, a matrix $A \in \mathbb{R}^{(n+1)\times(n+1)}$ and a vector $b \in \mathbb{R}^{n+1}$, if and only if M is one of the following hypersurfaces.*
(i) a hypersurface with zero $(k+1)$th mean curvature,
(ii) an open piece of $\mathbb{S}^n(r)$,
(iii) an open piece of $\mathbb{S}^m(c) \times \mathbb{R}^{n-m}$, with $k+1 \leq m \leq n-1$.

Theorem 3.2. Let $x : M_p^n \to \bar{M}_1^{n+1}$ be an isometric immersion satisfying the condition $\Delta x = Ax + b$. Then its mean curvature is constant.

Theorem 3.2 gives an key idea for the the following one.

Theorem 3.3. The only non k-minimal Lorentzian hypersurfaces in the Lorentz-Minkowski space-time, whose position vector field satisfies $L_k x = Ax + b$, are the isoparametric hypersurfaces.

Similarly, we can prove the following theorem.

Theorem 3.4. If a non k-minimal Lorentzian hypersurfaces in the Lorentz-Minkowski space-time satisfies the condition $L_k x = Ax + b$, Then, its shape operator is diagonal.

Theorem 3.5. Let $x : M_1^n \to \mathbb{L}^{n+1}$ be an orientable hypersurface immersed into the Lorentz-Minkowski space-time. Then, x satisfies $\Box\, x = Ax + b$, for an integer

$0 \leq k < n$, a matrix $A \in \mathbb{R}^{(n+1)\times(n+1)}$ and a vector $b \in \mathbb{R}^{n+1}$, if and only if M is one of the following hypersurfaces.
(i) a hypersurface with zero scalar curvature,
(ii) $\mathbb{S}^n_1(c)$, $c > 0$;
(ii) $\mathbb{S}^m_1(c) \times \mathbb{R}^{n-m}$, $c > 0$, and $1 < m < n$;
(iii) $\mathbb{S}^m(c) \times \mathbb{R}^{n-m}_1$, $c > 0$, and $1 < m < n$.

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ON STEIN MANIFOLDS AND BIHARMONIC REAL HYPERSURFACES

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Abstract. In this paper, we consider real hypersurfaces of Stein manifolds. A Stein manifold $(\mathbb{M}, J, \mathbf{g})$ is a complex manifold $\mathbb{M}$ with a complex structure J, a Kähler metric $\mathbf{g}$ and a fundamental form $\mu = i\partial\bar{\partial}\rho$, where $\rho : \mathbb{M} \to \mathbb{R}$ is a smooth strictly plurisubharmonic exhaustion. We study the biharmonicity condition on the submanifolds of Stein manifolds emphasizing on real hypersurfaces.

Key words and phrases: Plurisubharmonic;symplectic form; Stein manifold.

1. Introduction

A submanifold M^k of a Riemannian manifold $(\mathbb{M}^n, g)$ (where $1 \leq k \leq n$) defined by an isometric immersion $\mathbf{x} : M^k \to \mathbb{M}^n$ is called harmonic if it satisfies the condition $\tau(\mathbf{x}) = 0$, where the tension operator τ is the trace of $\nabla(d\mathbf{x})$ (i.e. $\tau(\mathbf{x}) := tr(\nabla d\mathbf{x})$). Here, ∇ denotes the Levi-Civita connection on M^k. As a routine extension, M^k is said to be biharmonic if it satisfies the Euler-Lagrange condition $\tau_2(\mathbf{x}) = 0$. The bitension operator τ_2 is defined by

$$\tau_2(\mathbf{x}) = \Delta\tau(\mathbf{x}) - tr(\bar{\mathrm{R}}(d\mathbf{x}, \tau(\mathbf{x}))d\mathbf{x}),$$

where, Δ and $\bar{\mathrm{R}}$ stand for the Laplace operator on M^k and the curvature tensor of $\mathbb{M}^n$, respectively, with the following rules.

$$\Delta V = tr(\nabla^2 V),$$
$$\bar{\mathrm{R}}(X, Y) = \left[\bar{\nabla}_X, \bar{\nabla}_Y\right] - \bar{\nabla}_{[X,Y]},$$

for every tangent vector fields V on M^k and X and Y on $\mathbb{M}^n$. The notation $\bar{\nabla}$ stands for the Levi-Civita connection on $\mathbb{M}^n$. In the compact case, every harmonic submanifold $\mathbf{x} : M^k \to \mathbb{M}^n$ plays the role of a critical point of the energy functional

$$\mathbf{e}(\mathbf{x}) = \frac{1}{2}\int_M |d\mathbf{x}|^2 ,$$

[1] and similarly, a biharmonic map has the role of a critical point of the map $\mathbf{e}_2$ (namely, the functionl of bienergy) defined by

$$\mathbf{e}_2(\mathbf{x}) = \frac{1}{2}\int_M |\tau(\mathbf{x})|^2 .$$

The variational problem associated to $\mathbf{e}_2$ is related to the tensor of stress-energy. In 1986, Jiang [3] has studied the formulae of the first and second variations of $\mathbf{e}_2$, which are used to define the biharmonic maps.

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It is well-known that every noncompact complete Riemannian manifold with positive sectional curvature is diffeomorphic to an Euclidean space. We note that such a manifold has a strictly convex exhaustion function. In complex case, every complex manifold which admits a strictly plurisubharmonic smooth exhaustion function is a Stein manifold. The class of Stein manifolds under the name of holomorphic complete manifolds has been firstly introduced (in 1951) by Karl Stein. The aim of this paper is to study the biharmonic real hypersurfaces of Stein manifolds.

2. Preliminaries

Here, some prerequisites are recalled from [2, 1, 4]. An *almost complex manifold* is a smooth real manifold $\mathbb{M}$ of real dimension $2n$ with an tangent bundle automorphism $J : T\mathbb{M} \to T\mathbb{M}$ satisfying $J^2 = -I$. When J satisfies the well-known *Nijenhuis identity* $N \equiv 0$, it is called *integrable* and then, $(\mathbb{M}, J)$ is called a *complex manifold*. Remember that

$$N(X,Y) := [JX, JY] - [X,Y] - J[X,JY] - J[JX,Y],$$

for all $X, Y \in T\mathbb{M}$.

By definition, a Kähler manifold $(\mathbb{M}, J, g)$ is a complex manifold $\mathbb{M}^k$ with a Hermitian metric $\mathbf{F} = \sigma - i\nu$ on tangent bundle such ν is a closed 2-form (i.e. $d\nu = 0$). The pair (ν, J) is said to be *compatible* if $\nu(., J.)$ is a Riemannian metric. So, on each almost complex submanifold M of $\mathbb{M}$ there is a symplectic form induced by ν which is compatible with $J|_M$. Remember that, each complex submanifold of a Kähler manifold is also a Kähler manifold. On any submanifold M of the complex manifold $\mathbb{M}$, there are some auxiliary operators as

$$\tilde{\phi} : TM \to TM, \quad \tilde{\psi} : TM \to NM, \quad \tilde{\tau} : NM \to TM, \quad \tilde{\eta} : NM \to NM,$$

defined by $X = \tilde{\varphi}X + \tilde{\psi}X$ and $JZ = \tilde{\tau}Z + \tilde{\eta}Z$ for every $X \in TM$ and $Z \in NM$. Since $\nu(JX, Y) = -\nu(X, JY)$ for all $X, Y \in TM$, the mentioned operators satisfy the following equalities

$$\begin{aligned} &\tilde{\varphi}^2 X + \tilde{\tau}\tilde{\psi}X = -X, \ \tilde{\eta}^2 Z + \tilde{\psi}\tilde{\tau}Z = -Z, \\ &\tilde{\varphi}\tilde{\tau}Z + \tilde{\tau}\tilde{\eta}Z = 0, \ \tilde{\psi}\tilde{\varphi}X + \tilde{\eta}\tilde{\psi}X = 0, \\ &g(\tilde{\psi}X, Z) = -g(X, \tilde{\tau}Z). \end{aligned} \tag{2.1}$$

Moreover, $\tilde{\varphi}$ and $\tilde{\eta}$ are skew-symmetric.

In the rest, we discuss on a particular class of complex manifolds, namely, *Stein manifolds*. Let $(\mathbb{M}^n, J)$ is a complex manifold and $K \subset \mathbb{M}^n$ is compact. The holomorphically convex hull of K is

$$hc(K) := \left\{ z \in \mathbb{M}^n \mid \forall \phi \in \mathcal{O}(\mathbb{M}^n), \ |\phi(z)| \leq sup_{w \in K} \phi(w) \right\}.$$

Here, $\mathcal{O}(\mathbb{M}^n)$ stands for the set of holomorphic functions on $\mathbb{M}^n$. Now, $(\mathbb{M}^n, J)$ is said to be *holomorphically convex* if there is no compact $K \subset \mathbb{M}^n$ with non-compact $hc(K)$.

Definition 2.1. A holomorphically convex complex manifold satisfying two following conditions is called a *Stein manifold*:

(1) Separation: For each distinct points $z_1, z_2 \in \mathbb{M}$, there is a $\phi \in \mathcal{O}(\mathbb{M}^n)$ such that $\phi(z_1) \neq \phi(z_2)$,

(2) Local coordinates: For every point $z \in \mathbb{M}^n$, there exist functions $\phi_1, \dots, \phi_k \in \mathcal{O}(\mathbb{M})$ whose differentials are $\mathbb{C}$-linearly independent at z.

Every noncompact Riemann surface and the Cartesian product of two Stein manifolds are Stein manifolds. In a Stein manifold $\mathbb{M}$, the closed complex submanifolds and the subsets of the form $\{z \in \mathbb{M} \,|\phi(z) \neq 0\}$ for some nonconstant $\phi \in \mathcal{O}(\mathbb{M})$ are Stein manifolds. Especially, the open subsets of $\mathbb{C}$ and the convex domains in $\mathbb{C}^n$ are Stein manifolds. But, compact complex manifolds aren't Stein.

Definition 2.2. A *holomorphy domain* is a nonempty open subset $\mathrm{U} \subset \mathbb{C}^n$ such that there is no nonempty open subsets $\mathrm{V} \subset \mathrm{U}$ and $\mathrm{W} \subset \mathbb{C}^n$ satisfying the following conditions.

(1) W is connected and $\mathrm{W} \subset \mathrm{U}$ dose not occurs,
(2) For every $\phi \in \mathcal{O}(\mathrm{U})$, there is a function $\psi \in \mathcal{O}(\mathrm{W})$ such that $\phi|_{\mathrm{V}} = \psi|_{\mathrm{V}}$.

In fact, a holomorphy domain is an open set such that there are some $\phi \in \mathcal{O}(\mathrm{U})$ without any holomorphic extension to a bigger set. Every holomorphy domain in $\mathbb{C}^n$ is a Stein manifold. For every Stein manifold $\mathbb{M}^n$, there exists a holomorphic embedding as $\Psi : \mathbb{M} \to \mathbb{C}^{2n+1}$. In the following, we present some other statements of the holomorphy domain.

First, we recall from the theory of complex functions that, a $\mathcal{C}^2$-function f on an open set $\mathrm{D} \subset \mathbb{C}$ is harmonic if $\Delta f = 4\frac{\partial^2 f}{\partial z \partial \bar{z}} = 0$ on D. In this context, there are several related definitions as follow.

Definition 2.3. (1) A real-valued $\mathcal{C}^2$-function f on an open set $\mathrm{D} \subset \mathbb{C}$ is said to be *subharmonic* if for any domain U with $\overline{\mathrm{U}} \subset \mathrm{D}$ and any continuous real function h on $\overline{\mathrm{U}}$, which is harmonic on U and satisfies $h \leq f$ on $\partial\mathrm{U}$, we have $h \leq f$ on U.
(2) A $\mathcal{C}^2$-function f on an open set $\mathrm{D} \subset \mathbb{C}^n$ is said to be *plurisubharmonic* if it satisfies

$$\sum_{j,k=1}^{n} \frac{\partial^2 f(z)}{\partial z_j \partial \bar{z}_k} w_j \bar{w}_k \geq 0,$$

for every $z \in \mathrm{D}$ and $w \in \mathbb{C}^n$. Moreover, if the mentioned hermitian form is positive definite, f is called *strictly plurisubharmonic*. In the general case, a real function f on $\mathrm{D} \subset \mathbb{C}^n$ is plurisubharmonic if it is upper semicontinuous and for every $z, w \in \mathbb{C}^n$, the function $h(s) := f(z + sw)$ is subharmonic.
(3) An open set $\mathrm{D} \subset \mathbb{C}^n$ is called *pseudoconvex* if there exists a plurisubharmonic function $f \in \mathcal{C}^1(\mathrm{D})$ such that $\mathrm{D}_r := \{z \in \mathrm{D} : f(z) < r\}$ is relatively compact in D for every $r \in \mathbb{R}$.

Recently, A. Tran has proved that if a complex manifold $\mathbb{M}^n$ admits a strictly plurisubharmonic function $f \in \mathcal{C}^\infty(\mathbb{M}^n)$ such that for every $r \in \mathbb{R}$, $\mathbb{M}_r := \{z \in \mathbb{M} : f(z) < r\}$ is relatively compact in $\mathbb{M}^n$, then it is a Stein manifold.

In a complex manifold $\mathbb{M}$, every sequence $\{K_j\}_{j=1}^{\infty}$ of compact subsets satisfying $\cup_j K_j = \mathbb{M}$ and $K_j \subset int(K_{j+1})$ (for all j) is called a *compact exhaustion.*

3. Main results

The following theorem formulates the biharmonicity of an arbitrary submanifold of a Stein manifold.

Theorem 3.1. Let M^m be a biharmonic submanifold of a Stein manifold $(\mathbb{M}^n, J)$ and B, A, $\mathbf{H}$ be the second fundamental form, the shape operator and the mean curvature vector field of M, respectively, and $\bar{R}$ is the curvature tensor of $(\mathbb{M}^n, J)$. Then, M satisfies the following equalities.

$$\operatorname{tr}(\nabla A_{\mathbf{H}}) + \operatorname{tr}(A_{\nabla^{\perp}_{\cdot}\mathbf{H}}(\cdot)) = \{\sum_{k=1}^{m} \bar{R}(\mathbf{H}, e_k)e_k\}^{\top},$$

$$\Delta^{\perp}\mathbf{H} + \operatorname{tr}B(A_{\mathbf{H}}(\cdot), \cdot) = \{\sum_{k=1}^{m} \bar{R}(\mathbf{H}, e_k)e_k\}^{\perp}, \tag{3.1}$$

with respect to a chosen geodesic frame field $\{e_k\}_{k=1}^m$ on a neighborhood of any point $p \in M^m$,

It is a well known fact that any complex submanifold of a Stein manifold is necessarily minimal. Hence, we considering only the biharmonic real submanifolds.

Let now $\mathbf{x} : M^m \to \mathbb{M}^n_c$ be the embedding of a real submanifold M of dimension m in $\mathbb{M}^n_c$. Then the bitension tensor becomes

$$\tau_2(\mathbf{x}) = -m\{\Delta\mathbf{H} - \frac{1}{4}cm\mathbf{H} + \frac{3}{4}cJ(JH)^{\top}\}, \tag{3.2}$$

where $\mathbf{H}$ denotes the mean curvature vector field, Δ is the rough Laplacian, and $\top$ stands for the tangent component to M^m.

The next two results are on real hypersurfaces of a Stein manifold.

Theorem 3.2. Let M be a proper biharmonic real hypersurface with constant mean curvature in a Stein manifold $(\mathbb{M}^n_c, J)$. Then its second fundamental form B satisfies the equality $|B|^2 = \frac{n+1}{2}c$.

Theorem 3.3. Let M^m be a biharmonic real submanifold of a Stein manifold $(\mathbb{M}^n_c, J)$ and B, A and $\mathbf{H}$ be the second fundamental form, the shape operator and the mean curvature vector field of M, respectively. If $J\mathbf{H}$ is assumed to be normal to M, then, M satisfies the following equalities.

$$\text{(i)} \ -\Delta^{\perp}\mathbf{H} + tr(B(\cdot, A_{\mathbf{H}}\cdot)) - \frac{1}{4}cm\mathbf{H} = 0,$$

$$\text{(ii)} \ m\ grad(|\mathbf{H}|^2) + 4tr(A_{\nabla^{\perp}\mathbf{H}}(\cdot)) = 0.$$

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ON WEAKLY BIHARMONIC HYPERSURFACES IN LORENTZIAN 5-SPACE FORMS

FIROOZ PASHAIE AND LEILA SHAHBAZ

Abstract. One of interesting subjects in differential geometry is the biharmonic Lorentzian hypersurfaces of Lorentz 5-space form. A Lorentzian hypersurface $\psi : M_1^4 \to \mathbb{M}_1^5(c)$ is said to be C-biharmonic if it satisfies the extended biharmonicity condition $\mathrm{C}^2\psi = 0$. C is the well-known Cheng-Yau operator. We study weakly C-biharmonic Lorentzian hypersurfaces of $\mathbb{M}_1^5(c)$ with at most two distinct principal curvatures and constant mean curvature.

Key words and phrases: 1-minimal; Biharmonic; de Sitter space.

1. Introduction

A well-known conjecture of Bang-Yen Chen states that each biharmonic submanifold of an Euclidean space is minimal. Chen himself has verified his conjecture on biharmonic surfaces in Euclidean 3-space. Also, it has been affirmed on hypersurfaces of Euclidean 4-space $\mathbb{E}^4$ in [3].

In [2], the conjecture has been confirmed on hypersurfaces with at most two distinct principal curvatures in n-dimensional Euclidean space $\mathbb{E}^n$. Euclidean cases have been studied in [1] and more others. Biharmonicity condition is defined based on the Laplace operator Δ. Replacing Δ by the Cheng-Yau map C, we study the weakly C-biharmonic Lorentzian hypersurfaces in the Lorentz 5-space forms with constant mean curvature. The operator C denoting the linear part of the first variation of the second mean curvature function is an extension of the Laplace operator which stands for the linear part of the first variation of the ordinary mean curvature function.

2. Preliminaries

The notations and formulae are recalled from [4, 5]. The semi-Euclidean m-space $\mathbb{E}^m_\xi$ of index $\xi = 1, 2$ is equipped with the product defined by

$$\langle \mathbf{v}, \mathbf{w} \rangle = -\sum_{i=1}^{\xi} v_i w_i + \sum_{i=\xi+1}^{m} v_i w_i,$$

for each vectors $\mathbf{v} = (v_1, \ldots, v_m)$ and $\mathbf{w} = (w_1, \ldots, w_m)$ in $\mathbb{E}^m$. In this talk, we deal with the 5-dimensional Lorentz space forms with the following common notation

$$\mathbb{M}_1^5(c) = \begin{cases} \mathbb{S}_1^5(r) & (\text{if } c = 1/r^2) \\ \mathbb{L}^5 = \mathbb{E}_1^5 & (\text{if } c = 0) \\ \mathbb{H}_1^5(-r) & (\text{if } c = -1/r^2), \end{cases}$$

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where, for $r > 0$, $\mathbb{S}^5_1(r) = \{\mathbf{v} \in \mathbb{E}^6_1 | \langle \mathbf{v}, \mathbf{v}\rangle = r^2\}$ denotes the 5-pseudosphere of radius r and curvature $1/r^2$, and $\mathbb{H}^5_1(-r) = \{\mathbf{v} \in \mathbb{E}^6_2 | \langle \mathbf{v}, \mathbf{v}\rangle = -r^2, v_1 > 0\}$ denotes the pseudo-hyperbolic 5-space of radius $-r$ and curvature $-1/r^2$. In the canonical cases $c = \pm 1$, we get the de Sitter 5-space $d\mathbb{S}^5 := \mathbb{S}^5_1(1)$ and anti de Sitter 5-space $Ad\mathbb{S}^5 = \mathbb{H}^5_1(-1)$.

Let M^4_1 be a Lorentzian (timelike) hypersurface of a canonical Lorentz 5-space form (i.e. $\mathbb{M}^5_1(c)$ for $c = 0, \pm 1$) defined by an isometric immersion $\mathbf{x} : M^4_1 \to \mathbb{M}^5_1(c)$. The set of all smooth tangent vector fields on M^4_1 is denoted by $\chi(M^4_1)$. The symbols ∇ and $\bar{\nabla}$ denote the Levi-Civita connections on M^4_1 and $\mathbb{M}^5_1(c)$, respectively. Also, ∇^0 denotes the Levi-Civita connection on $\mathbb{E}^6_\nu$ (for $\nu = 1, 2$). The Weingarten formula on M^4_1 is $\bar{\nabla}_V W = \nabla_V W + \langle \mathrm{S}V, W\rangle \mathbf{n}$, for each $V, W \in \chi(M^4_1)$, where S is the shape operator associated to a unit normal vector field $\mathbf{n}$ on M^4_1. Furthermore, in the case $|c| = 1$, $\mathbb{M}^5_1(c)$ is a 5-hyperquadric with the unit normal vector field $\mathbf{x}$ and the Gauss formula $\nabla^0_V W = \bar{\nabla}_V W - c\langle V, W\rangle \mathbf{x}$.

Associated to a basis chosen on M^4_1, the second fundamental form (shape operator) S has four different matrix forms [4, 5]. When the metric on M^4_1 has diagonal form $\mathcal{G}_1 := \mathrm{diag}[-1, 1, 1, 1]$, the shape operator S is of form $\mathcal{D}_1 = \mathrm{diag}[\lambda_1, \lambda_2, \lambda_3, \lambda_4]$ or

$$\mathcal{D}_2 = \mathrm{diag}[\begin{bmatrix} \lambda_1 & \lambda_2 \\ -\lambda_2 & \lambda_1 \end{bmatrix}, \lambda_3, \lambda_4],\ (\lambda_2 \neq 0).$$

In the case of non-diagonal metric $\mathcal{G}_2 = \mathrm{diag}[\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, 1, 1]$ the shape operator is of form

$$\mathcal{D}_3 = \mathrm{diag}[\begin{bmatrix} \lambda_1 + \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \lambda_1 - \frac{1}{2} \end{bmatrix}, \lambda_2, \lambda_3] \text{ or } \mathcal{D}_4 = \mathrm{diag}[\begin{bmatrix} \lambda_1 & 0 & \frac{\sqrt{2}}{2} \\ 0 & \lambda_1 & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & \lambda_1 \end{bmatrix}, \lambda_2].$$

When $S = \mathcal{D}_k$, we say that M^4_1 is a $\mathcal{D}_k$-*hypersurface*.

Definition 2.1. We define the ordered quadruple $\{\kappa_1; \kappa_2; \kappa_3; \kappa_4\}$ of principal curvatures as follows:

$$\{\kappa_1; \kappa_2; \kappa_3; \kappa_4\} = \begin{cases} \{\lambda_1; \lambda_2; \lambda_3; \lambda_4\} & (\text{if } S = \mathcal{D}_1) \\ \{\lambda_1 + i\lambda_2; \lambda_1 - i\lambda_2; \lambda_3; \lambda_4\} & (\text{if } S = \mathcal{D}_2) \\ \{\lambda_1; \lambda_1; \lambda_2; \lambda_3\} & (\text{if } S = \mathcal{D}_3) \\ \{\lambda_1; \lambda_1; \lambda_1; \lambda_2\} & (\text{if } S = \mathcal{D}_4). \end{cases}$$

The characteristic polynomial of S on M^4_1 is of the form $Q(t) = \sum_{j=0}^4 (-1)^j s_j t^{4-j}$, where, $s_0 := 1$, $s_i := \sum_{1 \le j_1 < \ldots < j_i \le 4} \kappa_{j_1} \ldots \kappa_{j_i}$ for $i = 1, 2, 3, 4$.

Definition 2.2. The *jth mean curvature* H_j of M^4_1 is defined by equation $\binom{4}{j} H_j = s_j$ (for $j = 1, 2, 3, 4$). In special case, H_1 is the ordinary mean curvature H. The second mean curvature H_2 and the normalized scalar curvature R satisfy the equality $H_2 := n(n-1)(1-R)$. If H_j is identically null, then M^4_1 is said to be $(j-1)$-*minimal*.

Definition 2.3. (i) A $\mathcal{D}_1$-hypersurface M^4_1 is said to be *isoparametric* if all of it's principal curvatures are constant.
(ii) For $k = 2, 3, 4$, a $\mathcal{D}_k$-hypersurface M^4_1 is isoparametric if the coefficients in the minimal polynomial of its shape operator are constant.

Remark 2.4. Here we recall Theorem 4.10 from [4], which assures us that there is no isoparametric timelike hypersurface of $\mathbb{M}^5_1(c)$ with complex principal curvatures.

Definition 2.5. *The* j*th Newton transformation* $N_j : \chi(M_1^4) \to \chi(M_1^4)$ *is inductively defined by*

$$N_0 = I, \ N_j = s_j I - S \circ N_{j-1}, \ (j = 1, 2, 3, 4),$$

where, I *is the identity map.*

Now, we introduce a notation as

$$\mu_{i;k} = \sum_{1 \le j_1 < \cdots < j_k \le 4; j_l \ne i} \kappa_{j_1} \cdots \kappa_{j_k}, \quad (i = 1, 2, 3, 4; \quad 1 \le k \le 3).$$

Definition 2.6. The *Cheng-Yau operator* on M_1^4, is defined by the formula $\mathrm{C}(f) := tr(\mathrm{N}_j \circ \nabla^2 f)$ for every $f \in \mathcal{C}^\infty(M_1^4)$, where $\langle \nabla^2 f(\mathrm{V}), \mathrm{W} \rangle = \langle \nabla_{\mathrm{V}} \nabla f, \mathrm{W} \rangle$ for every $\mathrm{V}, \mathrm{W} \in \chi(M_1^4)$.

Remark 2.7. In the special case, $\mathrm{C}(f)$ has the explicit expression

$$\mathrm{C}(f) = \sum_{i=1}^{4} \epsilon_i \mu_{i,1} (e_i e_i f - \nabla_{e_i} e_i f),$$

with respect to an orthonormal basis $\{e_1, \ldots, e_4\}$ of tangent space on a (local) coordinate system of hypersurface M_1^4 in $\mathbb{M}_1^5(c)$. Where, $\epsilon_1 = -1$ and $\epsilon_i = 1$ for $i = 2, 3, 4$.

For a Lorentzian hypersurface $\mathbf{x} : M_1^4 \to \mathbb{M}_1^5(c)$, with a chosen (local) unit normal vector field $\mathbf{n}$, for an arbitrary vector $\mathbf{a} \in \mathbb{L}^5$ we use the decomposition $\mathbf{a} = \mathbf{a}^T + \mathbf{a}^N$ where $\mathbf{a}^T \in TM$ is the tangential component of $\mathbf{a}$, $\mathbf{a}^N \perp TM$, and we have the following formulae.

$$\begin{aligned}
\mathrm{C}\mathbf{x} =& 12 H_2 \mathbf{n} - 12 c H_1 \mathbf{x}, \\
\mathrm{C}^2 \mathbf{x} =& 24 \left(\mathrm{N}_2 \nabla H_2 - c \mathrm{N}_1 \nabla H_1 - 9 H_2 \nabla H_2 \right) \\
&+ 12 \big[\mathrm{C} H_2 - 12 H_2 (2 H_1 H_2 - H_3) - 12 c H_1 H_2 \big] \mathbf{n} \\
&- 12 c [\mathrm{C} H_1 - 12 (H_2^2 + c H_1^2)] \mathbf{x}.
\end{aligned}$$

Definition 2.8. A hypersurface $\mathbf{x} : M_1^4 \to \mathbb{M}_1^5(c)$ is said to be C-*biharmonic* if it satisfies the condition $\mathrm{C}^2 \mathbf{x} = 0$. It is said to be *weakly* C-*biharmonic* if it satisfies the following conditions

$$\text{(2.1)} \qquad \begin{aligned}
&\text{(i) } \mathrm{N}_2 \nabla H_2 - c \mathrm{N}_1 \nabla H_1 = 9 H_2 \nabla H_2 \\
&\text{(ii) } \mathrm{C} H_2 = 12 H_2 (2 H_1 H_2 - H_3) + 12 c H_1 H_2.
\end{aligned}$$

3. Main results

First, we consider the Lorentzian hypersurfaces of type $\mathcal{D}_1$ in $\mathbb{M}_1^5(c)$ which have diagonal shape operator.

Lemma 3.1. On every weakly C-biharmonic hypersurface of $\mathbb{M}_1^5(c)$ with real principal curvatures of constant multiplicities, the distribution of the space of principal directions is completely integrable. In addition, if a principal curvature is of multiplicity greater than one, then its multiplicity is constant on each integral submanifold of the corresponding distribution.

Theorem 3.2. Every weakly C-biharmonic $\mathcal{D}_1$-hypersurface of $\mathbb{M}_1^5(c)$ with one principal curvature of multiplicity 4 is 1-minimal.

Proof. Let $\mathbf{x} : M_1^4 \to \mathbb{M}_1^5(c)$ be such a hypersurface with a principal curvature λ of multiplicity 4. Since $H_2 = \lambda^2$, it is enough to show that H_2 is constant on the open set $\mathcal{U} := \{p \in M_1^4 : \nabla H_2^2(p) \neq 0\}$. With respect to the basis $\{e_i| \ i = 1, 2, 3, 4\}$ as a local orthonormal frame of principal directions of the shape operator S on $\mathcal{U}$ such that for $i = 1, 2, 3, 4$ we have $\mathrm{S}e_i = \lambda e_i$ and

$$\mu_{i,2} = 3\lambda^2, \ H_2 = \lambda^2. \tag{3.1}$$

From condition (2.1)(i) and the polar decomposition $\nabla H_2 = \sum_{i=1}^4 \epsilon_i \langle \nabla H_2, e_i\rangle e_i$, we get

$$\epsilon_i\langle \nabla H_2, e_i\rangle(\mu_{i,2} - 9H_2) = 0,$$

on $\mathcal{U}$ for $i = 1, 2, 3, 4$. Hence, if for some i we have $< \nabla H_2, e_i >\neq 0$ on $\mathcal{U}$, then we get $\mu_{i,2} = 9H_2$ which, using equalities (3.1), gives $\lambda^2 = 0$ and then $H_2 = 0$ on $\mathcal{U}$, which is a contradiction. Hence $\mathcal{U}$ is empty and H_2 is constant on M. So, λ is constant.

Now, we show that $H_2 \equiv 0$. Having assumed that (locally) $H_2 \neq 0$, by (2.1)(ii), we obtain $\mathrm{C}(H_2) = 12H_2(2H_1H_2 - H_3) = 0$, which gives $2H_1H_2 - H_3 = 0$, then, $2\lambda^3 - \lambda^3 = 0$. Hence, again we get $\lambda = 0$ and then $H_2 = 0$, which is a contradiction. So $H_2 \equiv 0$. □

Theorem 3.3. Every weakly C-biharmonic $\mathcal{D}_1$-hypersurface of $\mathbb{M}_1^5(c)$ with exactly two distinct principal curvatures of multiplicities 3 and 1 and constant mean curvature is 1-minimal.

Theorem 3.4. Every weakly C-biconservative $\mathcal{D}_1$-hypersurface of $\mathbb{M}_1^5(c)$ with exactly two distinct principal curvatures of multiplicities 2 and constant mean curvature is 1-minimal.

Now, we consider the Lorentzian hypersurfaces of type $\mathcal{D}_2$, $\mathcal{D}_3$ and $\mathcal{D}_4$ in $\mathbb{M}_1^5(c)$ which have non-diagonal shape operator. The method of proofs are similar to the proof of Theorem 3.2.

Theorem 3.5. Let $\psi : M_1^4 \to \mathbb{M}_1^5(c)$ be a weakly C-biharmonic $\mathcal{D}_2$-hypersurface. If M_1^4 has constant mean curvature and at most two distinct principal curvatures, then it is 1-minimal.

Theorem 3.6. Every weakly C-biharmonic $\mathcal{D}_3$-hypersurface of $\mathbb{M}_1^5(c)$ with constant mean curvature and at most two distinct principal curvatures is 1-minimal.

Theorem 3.7. Let $\psi : M_1^4 \to \mathbb{M}_1^5(c)$ be a weakly C-biharmonic $\mathcal{D}_4$-hypersurface with constant mean curvature and at most two distinct principal curvatures. Then it is 1-minimal.

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2-CONFORMAL VECTOR FIELDS IN SOL SPACE

GHODRATALLAH FASIHI-RAMANDI AND FARZANEH SHAMKHALI

Abstract. In this paper, we introduce the concept of 2-conformal vector fields which are generalization of Killing and conformal vector fields on Reimannian and semi-Reimannain manifolds. Then, we characterize proper 2-conformal vector fields in Sol space.

Key words and phrases: Sol space; 2-conformal vector field; Reimannain geometry.

1. Introduction and preliminaries

The model space Sol in the sense of W. Thurston [2] is the Cartesian space $\mathbb{R}^3(x, y, z)$ equipped with a homogeneous metric (see [1])

$$g = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2.$$

The Sol space is a Lie group G with respect to multiplication law

$$(x, y, z) * (a, b, c) = (x + e^{-z}a, y + e^{z}b, z + c).$$

The left-invariant orthonormal frame field, i.e. the basis of the Sol space, is given by

$$e_1 = e^{-z}\partial_x, \quad e_2 = e^{z}\partial_y, \quad e_3 = \partial_z.$$

The Levi-Civita connection ∇ of Sol space is given by

$$\begin{aligned}
&\nabla_{e_1}e_1 = -e_3, \quad \nabla_{e_1}e_2 = 0, \quad \nabla_{e_1}e_3 = e_1,\\
&\nabla_{e_2}e_1 = 0, \quad \nabla_{e_2}e_2 = e_3, \quad \nabla_{e_2}e_3 = -e_2,\\
&\nabla_{e_3}e_1 = 0, \quad \nabla_{e_3}e_2 = 0, \quad \nabla_{e_3}e_3 = 0.
\end{aligned}$$

The non-vanishing components of the Reimannian curvature tensor are

$$R_{1212} = 1, \quad R_{1313} = -1, \quad R_{2323} = -1.$$

Then the Ricci tensor is given by $R_{11} = R_{22} = 0$ and $R_{33} = -2$. Hence the scalar curvature is -2.

2-conformal field. A vector field $\mathbf{Z} \in \mathcal{X}(M)$ is called 2-conformal vector field on a Riemannian manifold (M, g) if

$$L_{\mathbf{Z}}L_{\mathbf{Z}}g = 2\sigma g,$$

for some smooth function σ on M. Where, L is the Lie derivative operator on M. If σ is identically zero, the vector field $\mathbf{Z}$ is said to be 2-conformal vector field. Also, when σ be a constant the 2-conformal vector field $\mathbf{Z}$ is called homothetic and otherwise it said to be non-homothetic field. Obviously, every conformal vector field is 2-conformal vector field, hence, proper 2-conformal field is defined as a 2-conformal field which is not a conformal vector field.

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Theorem 1.1. *Let $\mathbf{Z} \in \mathcal{X}(M)$ be a 2-conformal vector field on a Riemannian manifold (M, g). Then*

$$g(\nabla_{\mathbf{Z}}\nabla_X\mathbf{Z} - \nabla_{[\mathbf{Z},X]}\mathbf{Z}, Y) + g(X, \nabla_{\mathbf{Z}}\nabla_Y\mathbf{Z} - \nabla_{[\mathbf{Z},Y]}\mathbf{Z}) + 2g(\nabla_X\mathbf{Z}, \nabla_Y\mathbf{Z}) = 2\sigma g(X, Y),$$

for any vector filed $X, Y \in \mathcal{X}(M)$. Here ∇ stands for the Levi-Civita connection of g.

The following result is quit direct and helpful in the sequel.

Corollary 1.2. *A vector field $\mathbf{Z} \in \mathcal{X}(M)$ is 2-conformal if and only if*

$$Rm(\mathbf{Z}, X, \mathbf{Z}, X) = g(\nabla_X\mathbf{Z}, \nabla_X\mathbf{Z}) + g(\nabla_X\nabla_{\mathbf{Z}}\mathbf{Z}, X) - \sigma g(\mathbf{Z}, X),$$

for any vector field $X \in \mathcal{X}(M)$, where Rm denotes the $(0, 4)$-type Reimannian curvature tensor of g.

The symmetry of the above formula, shows that $\mathbf{Z}$ is 2-conformal vector field if and only if

$$g(\nabla_X\mathbf{Z}, \nabla_X\mathbf{Z}) + g(\nabla_{\mathbf{Z}}\nabla_X\mathbf{Z} - \nabla_{[\mathbf{Z},X]}\mathbf{Z}, X) = \sigma g(\mathbf{Z}, X).$$

2. Main results

In this section we use the Corollary (1.2) to explore 2-conformal vector fields (further called 2-CVF) in Sol space.
Let assume that the 2-conformal vector field $\mathbf{Z}$ is given by

$$\mathbf{Z} = a(x, y, z)e_1 + b(x, y, z)e_2 + c(x, y, z)e_3.$$

After long but straightforward computation the following system of PDE's are implied

$$\begin{aligned} &(2c^2 + cc_z + a_yb_x + a_{xy}b) + e^{-z}(a_zc_x + 3a_xc + a_{xz}c) \\ &+ e^zbc_y + e^{2z}(2a_x^2 + aa_{xx} + b_x^2 + c_x^2) + \sigma a = 0, \end{aligned} \tag{2.1}$$

$$\begin{aligned} &(2c^2 - cc_z + a_yb_x + b_{xy}a) + e^{-z}(b_zc_y + 3b_yc - b_{yz}c) \\ &- e^{-z}ac_x + e^{2z}(2a_y^2 + bb_{yy} + 2b_y^2 + c_y^2) + \sigma b = 0, \end{aligned} \tag{2.2}$$

$$\begin{aligned} &\big((a - a_z)^2 + (b + b_z)^2 + 2c_z^2 + cc_{zz}\big) + e^z(bc_y + b_zc_y + bc_{yz}) \\ &+ e^{-z}(-ac_x + a_zc_x + ac_{xz}) + \sigma c = 0 \end{aligned} \tag{2.3}$$

Unfortunately, the 2-CVF is complicated nonlinear second order system of PDE's. Although we can't find exact solutions, we can determine 2-conformal vector fields that are generalization of fields ∂_x and ∂_y.

Let assume that $\mathbf{Z} = a(x, y, z)e_1$, i.e $b = c = 0$. Then the 2-CVF system became

$$e^{-2z}(2a_x^2 + aa_{xx}) + \sigma a = 0, \quad e^{2z}a_y^2 = 0, \quad (a - a_z)^2 = 0. \tag{2.4}$$

Form the second and the third equation of above, it follows $a(x, z) = f(x)e^z$, and from the first equation we get the differential equation

$$2f'^2 + ff'' + e^z\sigma f = 0,$$

which shows that $\sigma = g(x)e^{-z}$. Hence we have

$$2f'^2 + ff'' + g(x)f = 0.$$

Obviously, this ordinary differential equation has not elementary function solution for $f = f(x)$ in general. But, if $g(x) = \alpha$ for some non-zero constant α (to make sure our field in not 2-Killing), we can solve the equation analytically. In this case, we have

$$f(x) = c_1 \sqrt[5]{e^{\alpha x}} + c_2, \quad c_1, c_2 \in \mathbb{R}, \quad c_1 > 0.$$

Particulary for $x = 0$, $c_1 = 1$ and $c_2 = 0$, we get the field $X_1 = e^z e_1 = \partial_x$.

Quit analogously, assuming that $\mathbf{Z} = b(x, y, z)e_2$, i.e. $a = c = 0$, the 2-CVF system became

$$e^{-2z} b_x^2 = 0, \quad e^{-2z} 2b_y^2 + b b_{yy} = 0, \quad (b + b_z)^2 = 0.$$

The solution of this system is a function $b(y, z) = c_3 \sqrt[5]{e^{\alpha y}} + c_4$, and hence the 2-conformal vector field is $\mathbf{Z} = (c_3 \sqrt[5]{e^{\alpha y}} + c_4)e^{-z} e_2$. Particulary for $y = 0, c_3 = 1$ and $c_4 = 0$ we get the field $X_2 = e^{-z} e_2 = \partial_y$.

If we assume that $\mathbf{Z} = c(x, y, z)e_3$, i.e. $a = b = 0$, then the 2-CVF system became

$$2c^2 + c c_z + e^{-2z} c_x^2 = 0, \quad 2c^2 - c c_z + e^{2z} c_y^2 = 0, \quad 2c_z^2 + c c_{zz} + \alpha c = 0.$$

But one can check that this system has no solution except $c = 0$. Therefore, there is no 2-conformal vector field of form $\mathbf{Z} = c(x, y, z)e_3$.

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IS DARK MATTER AN EFFECT OF LORENTZIAN METRIC INDEX?

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Abstract. In this paper, we consider the Ricci soliton equation as a generalization of Einstein manifolds. If the potential vector field of our soliton be semi-Killing field, then deduce that it is related to the notion of so-called dark matter in general theory of relativity. To have a non-trivial such soliton, the metric of underlying manifold have to be Lorentzain. Hence, the dark matter and dark energy can be an effect of Lorentzian metric index.
Key words and phrases: Ricci Soliton; General Relativity; Dark matter.

1. Introduction

Ricci solitons are the natural generalization of Einstein metrics. A (semi-)Riemannian manifold (M, g) is said to be a Ricci soliton if there exists a vector field $X \in \mathcal{X}(M)$ and a real scalar λ, such that

$$\frac{1}{2}\mathcal{L}_X g + \mathrm{Ric} = \lambda g,$$

where $\mathcal{L}_X$ and Ric denote the Lie derivative in the direction of X, and the Ricci tensor, respectively.
It is called shrinking when $\lambda > 0$, steady when $\lambda = 0$, and expanding when $\lambda < 0$. If $X = \nabla f$ the equation can also be written as

$$\mathrm{Ric} + \mathrm{Hess} f = \lambda g,$$

and is called a gradient (Ricci) soliton. See [2, 3, 5] for background on Ricci solitons and their connection to the Ricci flow. We remark here that on a compact manifold Ricci solitons are always gradient solitons [12] and that every non-compact shrinking soliton is a gradient soliton [11].
During the last two decades, the geometry of Ricci solitons has been the focus of attention of many researchers. There are two aspects of the study of Ricci solitons, one looking at the influence on the topology by Ricci soliton (see e.g. [1, 4, 8, 10]) and the other looking at its influence on its geometry (see e.g. [7, 6, 9]). In this paper we are interested in the geometry of Ricci solitons arise frome a new geometric vector fields (called semi-Killing field) on semi-Riemannain manifolds. Then, the physical interpretation of our new structure will be presented.

Definition 1.1. *A vector field X on a semi-Riemannian manifold (M, g) is said to be a semi-Killing vector field, if $\mathcal{L}_X g = 2\alpha X^\flat \otimes X^\flat$ for a constant α where, $X^\flat$ is dual 1-form of X.*

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Clearly, the zero vector field $X = 0$ is a semi-Killing vector field and every Killing vector field X is semi-Killing with $\alpha = 0$. Also, we can construct a non-trivial semi-Killing vector field. Let $M = (a, b) \subset \mathbb{R}$ be an open interval and consider $g = ds^2$. Suppose that X is a nowhere zero semi-Killing vector field on M with some non-zero α. If $X^\flat = h(s)ds$, then the condition $\mathcal{L}_X g = 2\alpha X^\flat \otimes X^\flat$ lead us to the following ordinary differential equation

$$-2h'(s) = 2\alpha h^2(s),$$

and solving this equation gives $h(s) = \dfrac{1}{\alpha s + \beta}$ for some constant β.

Theorem 1.2. *Let X be a non-zero semi-Killing vector field on a closed (compact without boundary) Riemannain manifold (M, g). Then X is a Killing vector field.*

The above theorem shows that the set of semi-Killing vector fields on closed manifolds coincides with the set of all Killing vector field on them. Hence, existence of semi-Killing vector fields not only depends on the geometry of underlying manifold but also requires some topological constraints on the manifold.
Homogeneous spaces are among the nicest examples of Riemannian manifolds. In the following, we show that there is no non-trivial left-invariant semi-Killing vector field on a homogeneous manifold M with left-invariant metric g. Let X be a non-zero semi-Killing left invariant vector field on a homogeneous manifold (M, g).

Theorem 1.3. *Left-invariant semi-Killing vector fields on homogeneous spaces are Killing vector fields.*

2. Main results

In this section, prove our main results. First we prove that if (M^n, g, X, λ) be a Reimannain Ricci soliton with potential semi-Killing field, then X has to be a Killing vector field and (M, g) reduces to be an Einstein manifold.

Let (M, g, X, λ) is a Riemannain Ricci soliton with $\mathcal{L}_X g = 2\alpha X^\flat \otimes X^\flat$. Then, we have

$$\mathrm{Ric} = -2\alpha X^\flat \otimes X^\flat + \lambda g.$$

Tracing both sides of the above equation, we find $R = -2\alpha|X|^2 + n\lambda$, so by addition suitable expression to each side of the equation, we obtain

$$\mathrm{Ric} - \frac{1}{2}Rg + (\frac{n-2}{2})\lambda g = \alpha(|X|^2 g - 2X^\flat \otimes X^\flat).$$

As Einstein tensor $\mathrm{Ric} - \dfrac{1}{2}Rg$ is divergence free, so the right hand side of above equation must be divergence free.

Lemma 2.1. *Let X be a non-zero vector field on a Riemannian manifold (M, g). If divergence of symmetric tensor $T := |X|^2 g - 2X^\flat \otimes X^\flat$ vanishes, then* $\mathrm{div}(X) = 0$.

Now, with the assumptions of the previous Lemma, we can prove the following Theorem.

Theorem 2.2. *Riemannian Ricci solitons (M^n, g, X, λ) with $\mathcal{L}_X g = 2\alpha X^\flat \otimes X^\flat$, are Einstein manifold.*

Proof: Since (M^n, g, X, λ) is a Ricci soliton, we have

$$\mathrm{Ric} + 2\alpha X^\flat \otimes X^\flat = \lambda g.$$

If X be identically zero, then we have nothing to prove. Let X be a non-zero vector field, so Lemma 2.1 indicates that $\mathrm{div}(X) = 0$. On the other hand, we have

$$\mathcal{L}_X g = 2\alpha X^\flat \otimes X^\flat.$$

Tracing both sides of the above formula gives

$$\mathrm{div}(X) = \alpha |X|^2,$$

cosequently, $\alpha = 0$, and we have completed the proof.

The above theorem shows that there is no not-trivial Riemannian Ricci soliton with semi-Killing potential vector fields. Hence, we have to look for such structure in Lorentzian or other semi-Riemannain settings.

Theorem 2.3. *If (M^n, g, X, λ) be a Lorentz Ricci soliton with $\mathcal{L}_X g = 2\alpha X^\flat \otimes X^\flat$, then M has constant scalar curvature R.*

2.1. **Application to physics.** In this subsection, let (M^4, g, X, λ) is a Lorentz Ricci soliton which we regard it as a space-time manifold. Then, the Ricci soliton equation

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_X g = \lambda g,$$

become a generalization of Einstein field equation. In fact, tracing the both side of the above equation yields $R + \mathrm{div}(X) = 4\lambda$. the above equation can be rewritten as

$$\mathrm{Ric} - \frac{1}{2}Rg + \lambda g = \frac{1}{2}(\mathrm{div}(X)g - \mathcal{L}_X g).$$

In general theory of relativity, the scalar curvature R is related to distribution of mass in points of space-time, so regardless of λ which can be interpreted as cosmological constant, we may deduce that $\mathrm{div}(X)$ is related to notion of matter in space-time and $\frac{1}{2}(\mathrm{div}(X)g - \mathcal{L}_X g)$ is the momentum-energy tensor of this matter. Therefore, a Ricci soliton is a geometric structure which capable of describing matter and gravity, simultaneously.

However, the Ricci flow can be a framework for geometrization of matter in general relativity, it gives no more information about g as a potential for gravity and X as a potential for matter. Hence, it is natural to posing any other relation on X and g. If X be a Killing vector field, then the Ricci soliton equations coincides to Einstein equation in vacuum, and X gives the symmetries of this space-time. In this paper, we suggest X to satisfy the equation $\mathcal{L}_X g = 2\alpha X^\flat \otimes X^\flat$ for a non-zero constant α. Under this assumption, the Ricci soliton equation as a generalization of Einstein field equation, becomes

$$\mathrm{Ric} - \frac{1}{2}Rg + \lambda g = \alpha(\frac{|X|^2}{2}g - X^\flat \otimes X^\flat).$$

This equation shows that symmetric 2-tensor $T = |X|^2 g - X^\flat \otimes X^\flat$ must be divergence free. Applying this fact, a similar argument with Theorem 2.2 shows that X has to be a light-like vector field.

As we mentioned before, such structure in Riemannain settings lead to $X = 0$, and the structure reduces to Einstein manifold. But, as soon as we consider this structure in Lorentzian setting, we derive new field equation, with an internal relation between X and g. So, in our theory, X can be in related to the notion of dark matter in general relativity.

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$*$-CONFORMAL CURVATURE OF CONTACT METRIC MANIFOLDS

HANNANE FARAJI, BEHZAD NAJAFI, AND TAYEBEH TABATABAEIFAR

Abstract. We aim to introduce a new tensor, called $*$-conformal curvature tensor, in the contact manifolds. We provide $*$-conformal curvature tensor in Sasakian manifolds. Next, we deduce some properties of this tensor in Sasakian manifolds.

Key words and phrases: : $*$-conformal curvature; Sasakian manifolds.

1. Introduction

A differentiable manifold M^{2n+1} has an almost contact structure if it admits a 1-form η, a characteristic vector field ξ, and a (1,1) tensor field φ, which is satisfies

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{1.1}$$

where I indicates the identity endomorphism. Then, by (1.1), we can see that

$$\varphi\xi = 0, \quad \eta \circ \varphi = 0. \tag{1.2}$$

If an almost contact manifold M^{2n+1} admits a Riemannian metric g with the property

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{1.3}$$

for any vector fields $X, Y \in \chi(M)$. Then $(M^{2n+1}, g, \eta, \xi, \varphi)$ is called almost contact metric manifold (or for simplicity, ACM-manifold). An ACM-manifold is called normal if the (1,2)-type torsion tensor N_φ vanishes, where $N_\varphi = [\varphi, \varphi] + 2d\eta \otimes \xi$, where $[\varphi, \varphi]$ is the Nijenhuis tensor of φ. A normal ACM-manifold is called Sasakian manifold. A Sasakian manifold is also characterized by

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X,$$

for any vector fields $X, Y \in \chi(M)$. On a Sasakian manifold beside (1.1) and (1.3), we have

$$\nabla_X \xi = -\varphi X, \quad R(X, Y)\xi = \eta(Y)X - \eta(X)Y. \tag{1.4}$$

Several concepts in complex geometry have a counterpart in contact geometry. Tachibana introduces $*$-Ricci tensor on an almost Hermitian manifold. Afterward, Hamada defines the $*$-Ricci on the real hypersurface of a non-flat complex space form. This notion on an ACM-manifold $(M, g, \eta, \xi, \varphi)$ is defined as

$$^*Ric(X, Y) = \frac{1}{2} trace\Big\{Z \ \rightarrow R(X, \varphi Y)\varphi Z\Big\}, \quad \forall X, Y \in \chi(M). \tag{1.5}$$

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The $*$-Ricci operator ${}^*L$ is defined by $g({}^*LX, Y) = {}^*Ric(X, Y)$. With the help of $*$-Ricci tensor, several authors have investigated $*$-Ricci soliton in contact geometry (see [2]).

In a Riemannian manifold (M^{2n+1}, g), the conformal curvature tensor C is defined by

$$\begin{aligned} C(X,Y)Z =&K(X,Y)Z + \frac{r}{(2n)(2n-1)}\Big(g(Y,Z)X - g(X,Z)Y\Big)\\ &- \frac{1}{2n-1}\Big(Ric(Y,Z)X - Ric(X,Z)Y + g(Y,Z)LX - g(X,Z)LY\Big), \end{aligned} \tag{1.6}$$

where K denotes the curvature tensor of (1,3) type, Ric indicates the Ricci tensor, r is the scalar curvature, and L is the Ricci operator (M, g).

2. Main results

Definition 2.1. *In a contact metric manifold of dimension $2n+1$, the $*$-conformal curvature tensor is defined by*

$$\begin{aligned} {}^*C(X,Y)Z =&K(X,Y)Z + \frac{{}^*r}{2n(2n-1)}\Big(g(Y,Z)X - g(X,Z)Y\Big)\\ &- \frac{1}{2n-1}\Big({}^*Ric(Y,Z)X - {}^*Ric(X,Z)Y + g(Y,Z){}^*LX - g(X,Z){}^*LY\Big), \end{aligned} \tag{2.1}$$

where ${}^*r$ is the trace of the $*$-Ricci tensor, called $*$-scalar curvature.

Proposition 2.2. *In a contact metric manifold, the $*$-conformal curvature tensor obeys the following.*

(1) ${}^*C(X,Y)Z = -{}^*C(Y,X)Z$,
(2) ${}^*C(X,Y)Z + {}^*C(Y,Z)X + {}^*C(Z,X)Y = {}^*Ric(X,Y)Z + {}^*Ric(Y,Z)X + {}^*Ric(Z,X)Y$.

Definition 2.3. *A contact metric manifold is named ${}^*\eta$-Einstien if*

$${}^*Ric(X,Y) = ag(X,Y) + b\eta(X)\eta(Y),$$

where a, and b are smooth scalar functions on the manifold.

In [2], Ghash and Patra obtain the $*$-Ricci tensor in a $(2n+1)$-dimensional Sasakian manifold is as follows

$${}^*Ric(X,Y) = Ric(X,Y) - (2n-1)g(X,Y) - \eta(X)\eta(Y). \tag{2.2}$$

Theorem 2.4. *Suppose M^{2n+1} is a manifold with a Sasakian structure (g, η, ξ, φ). $(M^{2n+1}, g, \eta, \xi, \varphi)$ is an η-Einstien manifold if and only if it is an ${}^*\eta$-Einstien manifold.*

Proof. If $(M^{2n+1}, g, \eta, \xi, \varphi)$ is an η-Einstien manifold, then there are smooth scalar functions a and b on M, so that

$$Ric(X,Y) = ag(X,Y) + b\eta(X)\eta(Y). \tag{2.3}$$

From (2.2) and (2.3), we have

$${}^*Ric(X,Y) = \tilde{a}g(X,Y) + \tilde{b}\eta(X)\eta(Y), \tag{2.4}$$

where $\tilde{a} = a - (2n-1)$ and $\tilde{b} = b - 1$. So $(M^{2n+1}, g, \eta, \xi, \varphi)$ is a $^*\eta$-Einstien manifold. □

Equation (2.2) provides

$$^*LX = LX - (2n-1)X - \eta(X)\xi, \tag{2.5}$$

and

$$^*r = r - 4n^2, \tag{2.6}$$

where $^*Ric(X,Y) = g(^*LX, Y)$ and $^*r = trace\ of\ ^*L$. With the help of (2.2), (2.5) and (2.6) from (2.1), we get

$$\begin{aligned} ^*C(X,Y)Z =& C(X,Y)Z + \frac{1}{2n-1}\Big(\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y + g(Y,Z)\eta(X)\xi \\ & - g(X,Z)\eta(Y)\xi\Big) - \frac{2n-2}{2n-1}\Big(g(Y,Z)X - g(X,Z)Y\Big). \end{aligned} \tag{2.7}$$

Proposition 2.5. *In a Sasakian manifold, the $*$-conformal curvature tensor is given by* (2.7)*.*

Corollary 2.6. *In a Sasakian manifold, the $*$-conformal curvature tensor obeys the relation*

$$^*C(X,Y)Z + {}^*C(Y,Z)X + {}^*C(Z,X)Y = 0.$$

In a 3-dimensional manifold, C vanishes identically, and hence, we have

$$\begin{aligned} ^*C(X,Y)Z =& \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y \\ & + g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi. \end{aligned} \tag{2.8}$$

In this case, (2.8) infers $^*C$ does not vanish identically. Indeed, for any non-zero vector filed $\tilde{X}$ in the kernel of η, we have

$$^*C(2\tilde{X} + \xi, \tilde{X} + \xi)\xi = \tilde{X}.$$

Definition 2.7 ([4])**.** *A contact metric manifold is called ξ-conformally flat if* $C(X,Y)\xi = 0$.

Every Sasakian manifold becomes a K-contact manifold, but its inverse holds only in 3 dimensional. In [4], the authors prove that a K-contact manifold is ξ-conformally flat if and only if it is an η-Einstien Sasakian manifold.

Suppose a Sasakian manifold is ξ-conformally flat and ξ-$*$conformally flat, i.e., $^*C(X,Y)\xi = 0$. Then from (2.1), it follows

$$^*C(X,Y)\xi = C(X,Y)\xi + \frac{3-2n}{2n-1}\big[\eta(Y)X - \eta(X)Y\big]. \tag{2.9}$$

By hypothesis, $^*C(X,Y)\xi = 0 = C(X,Y)\xi$. Hence (2.9) and (1.4) turn into $R(X,Y)\xi = 0$.

Lemma 2.8 ([1])**.** *Let M^{2n+1} be a contact metric manifold that satisfies the relation $R(X,Y)\xi = 0$ for all X, Y. Then M is locally isometric to the Riemannian product of a flat $(n+1)$-dimensional manifold and an n-dimensional manifold of positive curvature 4, and flat for $n = 1$.*

Theorem 2.9. *If a Sasakian manifold M^{2n+1} is ξ-conformally flat along with ξ-$*$conformally flat, then the manifold M^{2n+1} is locally isometric to $\mathbb{E}^{n+1}(0)\times\mathbb{S}^n(4)$, $(n>1)$.*

If X is a conformal vector field, that is, $\mathcal{L}_X g = 2\rho g$ for some smooth function ρ, then it is known that $\mathcal{L}_X C = 0$. Hence

$$\begin{aligned}\mathcal{L}_X{}^*C(Y,Z)W =&\frac{1}{2n-1}\mathcal{L}_X\big[\eta(W)(\eta(Y)Z-\eta(Z)Y)-(g(Z,W)\eta(Y)-g(Y,W)\eta(Z))\xi\big]\\ &+\frac{2n-2}{2n-1}2\rho\big[g(Z,W)Y-g(Y,W)Z\big].\end{aligned} \tag{2.10}$$

According to S. Tanno and Blair [3] on an ACM-manifold, if there exists a vector field X obeying $\mathcal{L}_X\eta=\sigma\eta$ for certain function σ, X is called a contact vector field. Especially, X is called a strict infinitesimal contact transformation if $\sigma=0$.

Suppose the vector field X is a strict contact vector field and conformal vector field, then from (2.10), we get

$$\mathcal{L}_X{}^*C(Y,Z)W = 2\rho\frac{2n-2}{2n-1}\big[g(Z,W)Y-g(Y,W)Z\big], \tag{2.11}$$

where we have used $\mathcal{L}_X\xi^h=-2\rho\xi^h$. In particular, if $n=1$, then

$$\mathcal{L}_X{}^*C(Y,Z)W=0.$$

Lemma 2.10 ([3])**.** *If a vector field X satisfies $\mathcal{L}_X\eta=0$, then X satisfies $\mathcal{L}_X\xi=0$.*

By Lemma 2.10, one obtains the following corollary.

Corollary 2.11. *Let X be a conformal vector field with a strict infinitesimal contact transformation. then X is Killing.*

Proof. Suppose X is a strict infinitesimal contact transformation, i.e., $\mathcal{L}_X\eta=0$. Since X is a conformal vector field, we have $\mathcal{L}_X g=2\rho g$. Then by Lemma 2.10, we get $\mathcal{L}_X\xi=0$, which means that $\rho=0$. □

Theorem 2.12. *The $*$-Conformal tensor is invariant under the restricted contact and conformal vector fields on normal contact metric manifolds.*

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RICCI SOLITONS AND RICCI BI-CONFORMAL VECTOR FIELDS ON THE MODEL SPACE Sol_0^4

MAHIN SOHRABPOUR AND SHAHROUD AZAMI

Abstract. In the present paper, we classify the Ricci solitons and the Ricci bi-conformal vector fields on model space Sol_0^4. Also, we show that which of them are gradient vector fields and Killing vector fields.

Key words and phrases: Ricci bi-conformal vector fields; Ricci solitons; Killing field.

1. Introduction

Conformal vector fields play an important role in geometry and physics. In geometry, they are used to study conformal transformations and conformally invariant geometric quantities. In physics, they aries in theories with conformal symmetry, such as string theory and conformal field theory. Also, conformal vector fields preserve angles and ratios of distances between points on the manifolds.

A conformal vector field is a smooth vector field X on a Riemannian manifold (M, g) if a smooth function like f that named a potential function, exists on M that satisfies $\mathcal{L}_X g = fg$, where $\mathcal{L}_X g$ is the Lie derivative of g with respect X. So if the potential function $f = 0$, X is a Killing vector field. In other hand, if X is a non-Killing vector field, X is called to be a nontrivial conformal vector field. We say that X is a gradient conformal vector field, if X is a gradient of a smooth function. A conformal vector field explain completely in [4, 3]. At first, Garcia-Parrado and Senovilla introduced bi-conformal vector fields [7], then De et al. defined Ricci bi-conformal vector fields in [2]. If the following equations hold for some smooth functions α and β and any vector fields Y, Z, then the vector field X is called a Ricci bi-conformal vector field:

$$(\mathcal{L}_X g)(Y, Z) = \alpha g(Y, Z) + \beta S(Y, Z), \tag{1.1}$$

and

$$(\mathcal{L}_X S)(Y, Z) = \alpha S(Y, Z) + \beta g(Y, Z), \tag{1.2}$$

where S is the Ricci tensor of M.

One of the most important and attractive topics in physics and geometry is study of the Ricci solitons that are natural generalization of Einstein metrics. At first the Ricci soliton was introduced by Hamilton [8], have been studied in Lorentzian manifolds. On a pseudo-Riemannian manifold (M, g), it is defined by

$$\mathcal{L}_X g + S = \lambda g, \tag{1.3}$$

where X is a smooth vector field on M, and λ is a real number [1].

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If the group of isometries of (M,g) acts transitivity on M, the connected pseudo-Riemannian manifold (M,g) is named to be a homogeneous. Riemannian homogeneous spaces are a fundamental class of manifolds which study is common in geometry, algebra and group theory. A Thurston geometry (G,X) is a homogeneous space where X is connected and simply connected, let G be a group and G acts transitively on X with compact point stabilizers such that G is not contained in any larger group of diffeomorphisms of X, and there is at least one compact manifold modeled on (G,X). Thurston geometry is a subset of Riemannian homogeneous spaces, that studied in dimension three for three-manifolds. So the possible Riemannian structures of compact orientable three-manifolds are similar to the uniformization theorem for surfaces that are compact and orientable. We can decompose any three-manifold into pieces and each of them admits a Riemannian metric locally isometric to one of eight three-dimensional model spaces, the Thurston geometries $\mathbb{R}^3$, $\mathbb{S}^3$, $\mathbb{H}^3$, $\mathbb{S}^2\times\mathbb{R}$, $\mathbb{H}^2\times\mathbb{R}$, $\tilde{SL}(2,\mathbb{R}), Nil^3$ and Sol^3. Eight three-dimensional Thurston spaces explain completely in [9, 10]. The model space (Sol_0^4, g) is one of the four-dimensional Thurston geometries. Filipkiewicz in [6] listed 19 types of Thurston geometries in dimension four. According to Wall [11], the space (Sol_0^4, g) belongs to 14 spaces among these model spaces that admit complex structure compatible with the geometric structure, for more information study [5].

2. THE MODEL SPACE Sol_0^4

The primary manifold of the model space Sol_0^4 is $\mathbb{R}^4(x,y,z,t)$ with the group operation

$$(x_1,y_1,z_1,t_1)*(x_2,y_2,z_2,t_2)=(x_1+e^{t_1}x_2, y_1+e^{t_1}y_2, z_1+e^{-2t_1}z_2, t_1+t_2). \tag{2.1}$$

The left invariant Riemannian metric g of Sol_0^4 is obtained as follows

$$g=e^{-2t}(dx^2+dy^2)+e^{4t}dz^2+dt^2, \tag{2.2}$$

Therefore, we consider the metrically dual left invariant basis vector fields as

$$e_1=e^t\frac{\partial}{\partial x},\ \ e_2=e^t\frac{\partial}{\partial y},\ \ e_3=e^{-2t}\frac{\partial}{\partial z},\ \ e_4=\frac{\partial}{\partial t}. \tag{2.3}$$

So basis vector fields satisfy the following brackets.

$$\begin{aligned}&[e_1,e_2]=[e_1,e_3]=[e_2,e_3]=0, \quad [e_4,e_1]=e_1\\ &[e_4,e_2]=e_2, \quad [e_4,e_3]=-2e_3.\end{aligned}$$

The Levi-Civita connection of manifold (M,g) is shown by ∇. We can define the curvature tensor R of (M,g) as follows $R(X,Y)=\nabla_{[X,Y]}-[\nabla_X,\nabla_Y]$ and we define the Ricci tensor S by $S(X,Y)=tr(Z\to R(X,Z)Y)$. The components of Levi-Civita connection and Ricci tensor on Sol_0^4 are calculated by

$$\nabla_{e_i}e_j=\begin{pmatrix} e_4 & 0 & 0 & -e_1\\ 0 & e_4 & 0 & -e_2\\ 0 & 0 & -2e_4 & 2e_3\\ 0 & 0 & 0 & 0\end{pmatrix},\quad S=\begin{pmatrix}0&0&0&0\\0&-1&0&0\\0&0&-4&0\\0&0&0&6\end{pmatrix}. \tag{2.4}$$

For any vector field $X=X^ke_k$ by

$$(\mathcal{L}_Xg)(e_i,e_j)=X^kg(\nabla_{e_i}e_k,e_j)+e_iX^kg(e_k,e_j)+X^kg(e_i,\nabla_{e_j}e_k)+e_jX^kg(e_i,e_k),$$

we compute the Lie derivative of the metric g in direction to vector field X as follows

$$
\begin{aligned}
(\mathcal{L}_X g)_{11} &= -2X^4 + 2e_1X^1,\\
(\mathcal{L}_X g)_{12} &= e_1X^2 + e_2X^1,\\
(\mathcal{L}_X g)_{13} &= e_1X^3 + e_3X^1,\\
(\mathcal{L}_X g)_{14} &= X^1 + e_1X^4 + e_4X^1,\\
(\mathcal{L}_X g)_{22} &= 2e_2X^2 - 2X^4,\\
(\mathcal{L}_X g)_{23} &= e_2X^3 + e_3X^2,\\
(\mathcal{L}_X g)_{24} &= X^2 + e_2X^4 + e_4X^2,\\
(\mathcal{L}_X g)_{33} &= 2e_3X^3 + 4X^4,\\
(\mathcal{L}_X g)_{34} &= -2X^3 + e_3X^4 + e_4X^3,\\
(\mathcal{L}_X g)_{44} &= 2e_4X^4.
\end{aligned} \tag{2.5}
$$

Further, using the formula

$(\mathcal{L}_X S)(e_i, e_j) = -X^k S([e_k, e_i], e_j) + e_i X^k S(e_k, e_j) - X^k S(e_i, [e_k, e_j]) + e_j X^k S(e_i, e_k),$

the Lie derivative of the Ricci tensor in direction X is determined by

$$
\begin{aligned}
(\mathcal{L}_X S)_{11} &= 0,\\
(\mathcal{L}_X S)_{12} &= -e_1X^2,\\
(\mathcal{L}_X S)_{13} &= -4e_1X^3,\\
(\mathcal{L}_X S)_{14} &= 6e_1X^4,\\
(\mathcal{L}_X S)_{22} &= -2e_2X^2 + 2X^4,\\
(\mathcal{L}_X S)_{23} &= -e_3X^2 - 4e_2X^3,\\
(\mathcal{L}_X S)_{24} &= -X^2 + 6e_2X^4 - e_4X^2,\\
(\mathcal{L}_X S)_{33} &= -8e_3X^3 - 16X^4,\\
(\mathcal{L}_X S)_{34} &= 8X^3 - 4e_4X^3 + 6e_3X^4,\\
(\mathcal{L}_X S)_{44} &= 12e_4X^4.
\end{aligned} \tag{2.6}
$$

3. Ricci solitons and Ricci bi-conformal vector fields on the model space Sol_0^4

Now, we solve the equation (1.3) on the model space Sol_0^4. Substituting (2.4), (2.5), and (2.6) into (1.3), we get the following theorem.

Theorem 3.1. *The vector field X on (Sol_0^4, g) where g given by (2.2), is a Ricci soliton vector field if and only if*

$$X = (3x + b_2x + a_4)\frac{\partial}{\partial x} + (\frac{7}{2}y + b_2y + a_2)\frac{\partial}{\partial y} + (5z - 2b_2z + a_1)\frac{\partial}{\partial z} + b_2\frac{\partial}{\partial t}.$$

Now, we can investigate that which of Ricci solitons on (Sol_0^4, g) is as gradient vector field. Now, consider $X = \nabla f$ on (Sol_0^4, g) with potential function f.

Thus we have the following corollary.

Corollary 3.2. *There is not any gradient Ricci soliton X on (Sol_0^4, g).*

We solve the equation (1.1) and (1.2) on the model space Sol_0^4. Replacing (2.2), (2.4), (2.5), and (2.6) into (1.1) and (1.2), we get the following theorem.

Theorem 3.3. *The vector field X on (Sol_0^4, g) where g given by (2.2), is Ricci bi-conformal vector field if and only if*

$$X = (c_2x + c_4)\frac{\partial}{\partial x} + (c_2y + c_3)\frac{\partial}{\partial y} - 2c_2z\frac{\partial}{\partial z} + c_2\frac{\partial}{\partial t}.$$

Thus, we have the following theorem.

Theorem 3.4. *Any Ricci bi-conformal vector field X on (Sol_0^4, g) is gradient vector field with potential function f if and only if $f = c_6$.*

At the end we can state:

Corollary 3.5. *Any Ricci bi-conformal vector field X on (Sol_0^4, g) is Killing vector field.*

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ON ϕ-RECURRENT MIXED 3-STRUCTURES

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ABSTRACT. In the present paper, we study mixed 3-structure manifolds which their Riemannian curvature is ϕ-recurrent. In special case, we study mixed 3-cosymplectic manifolds and prove some results on 1-forms of the recurrent equation.

Key words and phrases: curvature; ϕ-recurrent; mixed 3-cosymplectic.

1. Introduction

Recurrent manifolds are generalization of locally symmetric manifolds. On locally symmetric manifolds covariant derivative of the Riemannian curvature is equal to 0. The sectional curvature of contact locally symmetric manifolds is constant which is a strong condition on the manifold. Thus Takahashi studied locally ϕ-symmetric manifolds [6] which satisfy $\phi(\nabla R) = 0$. As a generalization of all of the above concepts the notion of ϕ-recurrent manifolds have been introduced [3].

In this paper, we study ϕ-recurrent mixed 3-structure manifolds. We prove some properties of ϕ-recurrent mixed 3-cosymplectic manifolds and give an example.

2. Main results

Let M be an odd dimensional semi-Riemannian manifold such that there exist a vector field ξ, a 1-form η and a (1,1)-tensor field ϕ, on the manifold which the following condition hods

$$\phi^2 X = \epsilon(-X + \eta(X)\xi) \ , \ \eta(\xi) = 1 \ \forall X \in TM, \tag{2.1}$$

then (M, ξ, η, ϕ) is said to be an almost contact manifold for $\epsilon = 1$ and an almost para-contact manifold for $\epsilon = -1$ [1, 5].

Definition 2.1. *Let a semi-Riemannian manifold (M, g) have two almost para-contact structures (ξ_i, η_i, ϕ_i), $i = 1, 2$, and an almost contact structure (ξ_3, η_3, ϕ_3) such that satisfy as follows*

$$\eta_i(\xi_j) = 0, \ \phi_i(\xi_j) = \epsilon_j \xi_k, \ \phi_j(\xi_i) = -\epsilon_i \xi_k, \ \eta_i(\phi_j) = -\eta_j(\phi_i) = \epsilon_k \eta_k, \tag{2.2}$$

$$\phi_i o \phi_j - \epsilon_i \eta_j \otimes \xi_i = -\phi_j o \phi_i + \epsilon_j \eta_i \otimes \xi_j = \epsilon_k \phi_k, \tag{2.3}$$

$$g(\phi_i X, \phi_i Y) = \epsilon_i [g(X, Y) - \tau_i \eta_i(X) \eta_i(Y)], \ \forall X, Y \in TM, \tag{2.4}$$

in which (i, j, k) permutes for $(1, 2, 3)$ and $\tau_i = g(\xi_i, \xi_i) = \pm 1$.
Then $(M, \xi_i, \eta_i, \phi_i, g)_{i \in \{1,2,3\}}$ is said to be a metric mixed 3-structure manifold [4].

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In addition, a metric mixed 3-structure manifold $(M, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$ is called a mixed 3-cosymplectic manifold if

$$(\nabla_X \phi_i)Y = 0, \quad \forall X, Y \in TM, \quad i \in \{1, 2, 3\}. \tag{2.5}$$

Theorem 2.2 ([2]). *Let $(M, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$ be a mixed 3-cosymplectic manifold. Then the Ricci tensor of M is flat.*

Definition 2.3. *A metric mixed 3-structure manifold $(M, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$ is called a 3-ϕ-recurrent manifold, if the following condition holds*

$$\phi_i^2(\nabla_W R)(X, Y, Z) = A_i(W)R(X, Y)Z, \quad i = 1, 2, 3. \tag{2.6}$$

Here A_is are 1-forms on the manifold.

In the rest of the paper, we suppose the manifold admits mixed 3-cosymplectic structure.

Lemma 2.4. *On a mixed 3-cosymplectic manifold $(M, \xi_i, \eta_i, \phi_i)_{i\in\{1,2,3\}}$, the tensor fields ϕ_i's satisfy*

$$\phi_i^2 o \phi_j^2 = -\epsilon_k[\epsilon_i\phi_i^2 + \eta_j \otimes \xi_j] = -\epsilon_k[\epsilon_j\phi_j^2 + \eta_i \otimes \xi_i], \tag{2.7}$$

in which $\epsilon_1 = \epsilon_2 = -\epsilon_3 = -1$.

Theorem 2.5. *On a non-flat 3-ϕ-recurrent mixed 3-cosymplectic manifold $(M, \xi_i, \eta_i, \phi_i)$, $i \in \{1, 2, 3\}$, the 1-forms A_j and A_i satisfy in the following condition*

$$A_j(W) = \epsilon_k A_i(W), \ \forall W \in TM,$$

where $\epsilon_1 = \epsilon_2 = -\epsilon_3 = -1$ and $\{i, j, k\} = \{1, 2, 3\}$.

Proof. Since $(M, \xi_i, \eta_i, \phi_i)$, $i \in \{1, 2, 3\}$ is a mixed 3-cosymplectic manifold, we apply ϕ_i^2 on Equation (2.6) and use Lemma 2.4 and obtain

$$-\epsilon_k[\epsilon_i\phi_i^2(\nabla_W R)(X, Y, Z) + \eta_j((\nabla_W R)(X, Y, Z))\xi_j] =$$

$$\epsilon_i A_j(W)[-R(X, Y)Z + \eta_i(R(X, Y)Z)\xi_i]. \tag{2.8}$$

Hence we have,

$$\begin{aligned}(\epsilon_i A_j(W) - \epsilon_k\epsilon_i A_i(W))R(X, Y)Z &= \epsilon_k\eta_j((\nabla_W R)(X, Y, Z))\xi_j \\ &+ \epsilon_i\eta_i(A_j(W)R(X, Y)Z)\xi_i.\end{aligned} \tag{2.9}$$

By applying ϕ_j and then ϕ_k and η_k on (2.9) and by some computations we get

$$(\epsilon_i A_j(W) - \epsilon_k\epsilon_i A_i(W))\phi_k o \phi_j(R(X, Y)Z) = 0. \tag{2.10}$$

and so,

$$R(X, Y)Z = \eta_k(R(X, Y)Z)\xi_k. \tag{2.11}$$

But the previous equation implies $R = 0$ which is a contradiction because the manifold is non-flat, thus, the Equation (2.10) implies $A_j(W) = \epsilon_k A_i(W)$. □

By using Theorem 2.2, we have the following theorem.

Theorem 2.6. *Let $(M, \xi_i, \eta_i, \phi_i, g)_{i\in\{1,2,3\}}$ be a 3-ϕ-recurrent mixed 3-cosymplectic manifold. Then the covariant derivative of the Riemannian curvature of M vanishes.*

Example 2.7. *We suppose* $M = \left\{ (x_i)_{i=\overline{1,11}} \in \mathbb{R}^{11} | \sum_{i=5}^{8} x_i \neq 0 \right\}$, *and put* ϕ_i*'s as follows*

$$\begin{aligned} \phi_1((x_i)_{i=\overline{1,11}}) &= (-x_2, -x_1, -x_4, -x_3, \ldots, 0, x_{11}, x_{10}), \\ \phi_2((x_i)_{i=\overline{1,1}}) &= (x_4, -x_3, -x_2, x_1, \ldots, -x_{10}, -x_9, 0), \\ \phi_3((x_i)_{i=\overline{1,11}}) &= (x_3, -x_4, -x_1, x_2, \ldots, x_{11}, 0, -x_9). \end{aligned}$$

Now we define $f = x_5 + x_6 + x_7 + x_8$ *and*

$$g = \sum_{i=1}^{4} (-1)^i dx_i dx_i + \sum_{i=5}^{8} (-1)^i f^2 dx_i dx_i + dx_9 dx_9 - dx_{10} dx_{10} + dx_{11} dx_{11}.$$

The structure vector fields are defined as $\xi_1 = \partial x_9$, $\xi_2 = \partial x_{11}$, $\xi_3 = -\partial x_{10}$ *and* η_i*'s are dual of them. One can show that* $(M, \xi_i, \eta_i, \phi_i, g)_{i \in \{1,2,3\}}$ *is a metric mixed 3-structure manifold and curvature tensor* R *and its covariant derivative have the following non-zero components*

$$\begin{aligned} R_{rssr} = -R_{ijji} &= 4, \ i \neq j, r \neq s, \quad i, j \in \{5, 7\}, \quad r, s \in \{6, 8\}, \\ -R_{riis} = -R_{jiir} &= R_{irrj} = R_{srri} = 2, \\ R_{rssr,k} = -R_{ijji,k} &= \frac{-24}{f}, \\ R_{riis;k} = R_{jiir;k} &= -R_{irrj;k} = -R_{srri;k} = \frac{12}{f}, \end{aligned}$$

for $i \neq j, r \neq s$, $i, j \in \{5, 7\}, r, s \in \{6, 8\}$ *and* $k = 5, \ldots, 8$.

Therefore, the one form of the recurrent condition can be taken as

$$A(\partial x_k) = \begin{cases} \frac{-6}{f}, & k = 5, \ldots, 8; \\ 0, & k = 1 \ldots 4, and\ k = 9, 10, 11. \end{cases}$$

So, for any $X, Y, Z, W \in TM$, *we can write*

$$\phi_i^2 (\nabla_W R)(X, Y, Z) = A_i(W) R(X, Y, Z, U), \ i = 1, 2, 3,$$

where $A_1(W) = A_2(W) = -A_3(W) = A(W)$ *which means* M *is a 3-*ϕ*-recurrent manifold.*

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CONTACT HYPERSURFACE IN NEARLY KÄHLER MANIFOLDS

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Abstract. In this article, we will give a sufficient and necessary condition that a hypersurface of nearly Kähler manifolds with induced almost contact structure must be a contact hypersurface. Also, we show that such hypersurface can not be totally geodesic or totally umbilic.

Key words and phrases: Contact hypersurface; Nearly Kähler manifolds; almost contact structure.

1. Introduction

An almost complex manifold (M, J) (where J is (1,1)-form and $J^2 = -Id$) is called almost Hermitian there exist Riemannain metric like g such these metric is compatible with J. this means that $J^*g = g$. An almost Hermitian manifold (M, g, J) is called nearly Kähler manifold, if its Levi-Civita connection ∇ satisfies $(\nabla_X J)Y + (\nabla_Y J)X = 0$. These manifolds appear moreover in a natural way in the GrayHervella classification [4] as one class of the 16 classes of almost Hermitian manifolds.Recent interest in nearly Khler manifolds came from the fact, that in dimension 6 these manifolds are related to the existence of Killing spinors and that they admit a Hermitian connection with totally skew-symmetric torsion. Such connections are of interest in string theory [3].
In 2002, Nagy [5] proved that every simply connected complete nearly Khler manifold is isometric to a Riemannain product space of 3-classes naturally reductive 3-symmetric spaces 2- twister spaces over positive Quaternion-Khler manifolds and 3- six dimensional nearly Khler manifolds and Butruile in 2008 [1] showed that there exist only four complete,homogeneous 6-dimensional nearly Khler manifolds (up to homothety and covering space) and all of them are 3-symmetric.An important question in nearly Khler geometry is the fundamental explanation of rareness of such manifolds or difficulties of introducing non-homogeneous examples. Recently Foscolo and Haskin [2] proved that there exist new non-homogeneous Nearly Kähler structure on S^6 and $S^3 \times S^3$. they also shown that this non-homogeneous example of co-homogeneity one and conjectured that the all co-homogeneity one example are listed as non-homogeneous Nearly Kähler structure on S^6 and $S^3 \times S^3$. therefore the study of hypersurface of 6-Nearly Kähler manifold can be important. In this paper we study hypersurface of nearly Kähler manifolds with induced almost contact structure and investigate for a sufficient and necessary condition that such hypersurface must be a contact hypersurface also we prove that this hypersurface can not be totally geodesic or totally umbilic.

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2. Main results

Lemma 2.1 ([3]). *on Nearly kähler manifold (M, J, g) there exists a unique connection ∇ with totally skew-symmetric torsion T^{∇} satisfying $\nabla g = 0$ and $\nabla J = 0$. More precisely*

$$\nabla = D - \frac{1}{2}J\Sigma, \quad T^{\nabla} = -Jo\Sigma, \tag{2.1}$$

where D is Levi-Civita connection , $\Sigma = DJ$ and $\{\Sigma_X, J\} = 0$, for all vector fields X.

tensor field Σ of type (2, 1) has the properties as follows.

$$\Sigma(X, Y) = -\Sigma(Y, x), \tag{2.2}$$

$$\Sigma(X, JY) = -J\Sigma(X, Y), \tag{2.3}$$

$$g(\Sigma(X, Y), Z) = g(\Sigma(Y, Z), Z) = g(\Sigma(Z, X), Y), \tag{2.4}$$

let $f : P \longrightarrow M$ be an orientable hypersurface of the nearly Kähler manifold (M, J, g). If $\tilde{D}$ denote the Riemannian connection induced on P, then the Gauss and Weingarten formulas are

$$D = \tilde{D} + g(SX, Y)N, \quad SX = -D_X N, \tag{2.5}$$

where $X, Y \in TM$ and $N \in T^{\perp}M$ and S is the shape operator of the hypersurface P.

Definition 2.2. *A $(2n-1)$-dimensional smooth manifold M is said to be an almost contact metric manifold if carries a global 1- form η, a vector field ξ , a $(1, 1)$-tensor field Φ, and a Riemannain metric g satisfying*

$$\Phi^2 = -Id + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad g(\Phi X, \Phi Y) = g(X, Y) - \eta(X)\eta(Y). \tag{2.6}$$

On the other hand $\Phi(\xi) = 0$, $\eta o\Phi = 0$, $g(X, \xi) = \eta(X)$, and ξ is a unit vector field. The almost contact metric structure (η, ξ, Φ, g) on M is called a contact metric structure if $d\eta(X, Y) = g(X, \Phi Y)$.

On orientable hypersurface $f : P \longrightarrow M$ of the nearly Kähler manifold M we can define 3 tensor field as follow:

$$\Phi_1 X = JX - \eta(X)N, \quad \Phi_2 X = \sqrt{3}\Sigma(X, N), \quad \Phi_3 X = \sqrt{3}\Sigma(X, \xi), \tag{2.7}$$

where N is unit vector field normal to the hypersurface along f and $\xi = -JN$.

Proposition 2.3. *On orientable hypersurface $f : P \longrightarrow M$ of the nearly Kähler manifold M if η is the smooth 1-form dual to ξ then above tensor fields (ξ, η, Φ_i) for $i = 1, 2, 3$ are almost contact metric structures on P and defined a quaternion fields.*

At now we try to find the covariant derivatives of the structure tensor fields of the almost contact metric structures on the hypersurface P. with use fact that $\nabla J = 0$ we have $J\nabla_x N = \nabla_X JN$ and

$$D_X JN - 1/2J\Sigma(X, JN) = J(D_X N - 1/2J\Sigma(X, N)),$$

therefore

$$-D_X \xi - 1/2\Sigma(X, N) = -JSX + 1/2\Sigma(X, N),$$

and

$$D_X\xi = JSX - \Sigma(X,N) = \Phi_1(SX) + \eta(SX)N - \frac{\sqrt{3}}{3}\Phi_2.$$

Tangent part of above relation is $\tilde{D}_X\xi = \Phi_1(SX) - \frac{\sqrt{3}}{3}\Phi_2$. with use again of $\nabla J = 0$ and Gauss- Weingarten formulas

$$\nabla_X JY = D_X JY - 1/2J\Sigma(X,JY) = D_X(\Phi_1 Y) + X\eta(Y)N + \eta(Y)D_X N - 1/2\Sigma(X,Y),$$

$$J\nabla_X Y = J(D_X Y - 1/2J\Sigma(X,Y)) = J(\tilde{D}_X Y + g(SX,Y)N) + 1/2\Sigma(X,Y),$$

therefore

$$\begin{aligned} D_X(\Phi_1 Y) + X\eta(Y)N - 1/2\Sigma(X,Y) - \eta(Y)SX &= D_X(\Phi_1 Y) + X\eta(Y)N - 1/2\Sigma(X,Y) \\ &= \Phi_1(\tilde{D}_X Y) + \eta(\tilde{D}_X)N - g(SX,Y)\xi + 1/2\Sigma(X,Y), \end{aligned}$$

tangent part of this relation

$$(\tilde{D}_X\Phi_1)Y = \eta(Y)SX - g(SX,Y)\xi + \Sigma(X,Y)^T.$$

Denote by ω_i the fundamental 2-form on P given by $\omega_i(X,Y) = g(\Phi_i X,Y)$, for i=1,2,3.

Proposition 2.4. *The fundamental 2-form ω on a real hypersurface in a strictly Kähler manifold is not closed.*

Proof.

$$\begin{aligned} d\omega_1(X,Y,Z) &= X(\omega_1(Y,Z)) + Y(\omega_1(Z,X)) + Z(\omega_1(Y,Z)) \\ &= \tilde{g}((\tilde{D}_X\Phi_1)Y,Z) + \tilde{g}((\tilde{D}_X\Phi_1)Y,Z) + \tilde{g}((\tilde{D}_X\Phi_1)Y,Z) \\ &= \sigma_{X,Y,Z}\{\eta(Y)g(SX,Z) - g(SX,Y)\eta(Z) + g(\Sigma(X,Y),Z)\} \\ &= g(\Sigma(X,Y),Z) + g(\Sigma(Y,Z),X) + g(\Sigma(Z,X),Y) \\ &= 3g(\Sigma(X,Y),Z). \end{aligned}$$

□

Motivated by the example $S^{2n-1} \in \mathbb{C}^n$ Okumura in [6] called that P is a contact hypersurface of M if there exists an everywhere nonzero smooth function ρ on P such that $d\eta = 2\rho\omega$ holds on P along f. It is clear that if this equation holds, then then the rank of Φ being $2n-2$ and we have $\eta \wedge \Phi^{n-1} = \rho^{-n}\eta \wedge (d\eta)^{n-1} \neq 0$, that is, every contact hypersurface of a Nearly Kähler manifold is a contact manifold.

Theorem 2.5. *let $f : P \longrightarrow M$ be a real hypersurface of nearly Kähler manifolds M. then P is contact hypersurface under contact structure Phi_1 if there exist a non-where zero function ρ on P along f such that*

$$S\Phi_1 + \Phi_1 S = 2\rho\Phi_1 + 2\frac{\sqrt{3}}{3}\Phi_2.$$

Proof.

$$\begin{aligned} d\eta(X,Y) &= d\eta(Y)X - d\eta(X)Y - \eta([X,Y]) = g(Y,\nabla_X\phi) - g(X,\nabla_Y\phi) \\ &= g(Y,\Phi_1 SX - \frac{\sqrt{3}}{3}\Phi_2 X) - g(X,\Phi_1 SY - \frac{\sqrt{3}}{3}\Phi_2 Y) \\ &= g(Y,\Phi_1 SX) + g(Y,S\Phi_1 X) - 2\frac{\sqrt{3}}{3}g(Y,\Phi_2 X), \end{aligned}$$

from $d\eta(X,Y) = 2\rho\omega_1(X,Y) = 2\rho g(\Phi_1 X, Y)$ we have the result. □

Corollary 2.6. *let $f : P \longrightarrow M$ be a real contact hypersurface of nearly Kähler manifolds M under contact structure Φ_1. then P can note be totally geodesic or totally umbilic submanifolds of M*

Proof. from above theorem and relation $S\Phi_1 + \Phi_1 S = 2\rho\Phi_1 + 2\frac{\sqrt{3}}{3}\Phi_2$ if P is totally geodesic or totally umbilic then Φ_1 and Φ_2 must be linear dependent this means that there exist function Λ such that $\Phi_1 = \lambda\Phi_2$.

$$-I + \eta \otimes \xi = \Phi_1(\Phi_1) = \lambda\Phi_1(\Phi_2) = \lambda\Phi_3,$$

with chose a tangent vector X such that $Phi_3 X \neq 0$ we have

$$-x + \eta(X)\xi = \Phi_3(X).$$

But in not possible because (η, ξ, Φ_3) is an almost contact structure and

$$g(\Phi_3 X, x) = g(\Phi_3 X, \xi) = 0,$$

and its contraction with chose of X. □

Corollary 2.7. *let $f : P \longrightarrow M^n$ be a real contact hypersurface of nearly Kähler manifolds M under contact structure Φ_1. if X be a principle direction of hypersurface then $\Phi_1 X$ can not be a principle direction. therefore the maximal principle curvature of P is $\frac{n+1}{2}$.*

Remark 2.8. *In theorem 2.5 if M is Kähler manifolds and P is totally umbilic hypersurface in M such that the mean curvature of hypersurface in non-zero then P is contact hypersurface (in this case $\Phi_2 = \Phi_3 = 0$). therefore under contact structure Φ_1 induced form complex plane $\mathbb{C}^n$ every odd dimensional sphere is contact hypersurface. but in nearly Käher case S^5 under contact structure Φ_1 induced form S^6 (as totally geodesic hypersurface) is not contact hypersurface.*

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A KAKEYA PROBLEM ON RIEMANNIAN MANIFOLDS OF NONPOSITIVE CURVATURE

REZA MIRZAIE AND OMID REZAIE

Abstract. We generalize the definition of Kakeya set to Hadamard manifolds and we find a Kakeya set in hyperbolic space.

Key words and phrases: Fractal; Hyperbolic space; Kakeya problem.

1. Introduction

The Kakeya needle problem posed by S. Kakeya in 1917 asks whether there is a minimum area region (called Kakeya set), in the plane, in which a needle of length one can be turned through 360 degree continuously, and return to its initial position. Pal [7], showed that the solution of Kakeya‘s problem for convex sets is the equilateral triangle of height one. For the general case, when the Kakeya set is not necessarily convex or even simply connected, the answer was surprising. Besicovitch gave the answer that one could rotate a needle using an arbitrary small area. The Kakeya problem, can be mentioned a little more different such that the needle (line segment) can be replaced by line. So the problem is to find a planar domain with the smallest area so that a line segment can be rotated by 180 degrees in this domain. It is shown by A. S. Besicovitch [1], that for $n \geq 2$ there are subsets of R^n of measure zero which contain a line segment in each direction. Such sets are called Besicovitch sets or Kakeya sets. The Kakeya conjecture is that Kakeya sets in R^n must have Hausdorff dimension at least n. It is already proved for $n = 2$ but is still open in higher dimensions. The original construction of Kakeya set by Besicovitch has been simplified by other mathematicians (see [3, 4, 6]). Recently, the problem has received considerable attention due to its many applications. There are strong connections between Kakeya-type problems and problems in number theory [2], geometric combinatorics [9], arithmetic combinatorics, oscillatory integrals, and the analysis of wave equations [8]. Two main problems related to the original Kakeya are: First, trying to solve the Kakeya conjecture and the second one is to consider similar problem in more general cases and study the existence of Kakeya sets. We consider in this note the second problem. We can mention many problems similar to the Kakeya set problems. For example, in a similar problem it is shown that there are thin sets of circles sets of measure zero which contain circles of every radius. Instead of a plane, we can consider an sphere and the rotation takes place on the surface of the unit sphere and arc of great circle plays the role of the line. It is known that for arcs of length small compared to the radius of the sphere the answer is similar to the original Kakeya problem. A more general case is to replace the plane by a two dimensional Riemannian manifold in which the geodesics play the role of lines. Since direction of the geodesics is not meaningful in general case, it

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is necessary to give a new definition of the Kakeya set which generalizes the original definition on the plan sets. We give in the present article a definition for Kakeya set on Hadamard manifold which is a generalization of its original definition. Then, we find a Kakeya set in two dimensional hyperbolic space which is the most important model of a Hadamard manifold.

2. Main results

A point b in infinity of R^2 is by definition the collection of all lines in R^2 parallel to a line L and it is said that L passes from b. So, as Besicovitch's result, there is a Kakeya set K in R^2 such that for each point b at infinity, there is a line in K passing from b. Thus, the following definition 2.1, of the Kakeya set in Hadamard manifolds is a generalization of the original definition of the Kakeya set. First we recall some preliminary facts and definitions about Hadamard manifolds (see [5] for details). Let M be a simply connected Riemannian manifold of nonpositive curvature, which is called a Hadamard manifold. Two unit speed geodesics γ and β in M is said to be asymptotic if there exists a positive number c such that for all $t \geq 0$, one of the following is true

$$d(\gamma(t), \beta(t)) < c \quad or \quad d(\gamma(t), \beta(-t)) < c.$$

Asymptotic relation is an equivalence relation on all geodesics of M. The asymptotic class of a geodesic γ is denoted by $[\gamma]$. The collection of all asymptotic classes is called the infinity of M denoted by $M(\infty)$. So each point b in $M(\infty)$ has interpretation as a class of asymptotic geodesics. If $b = [\gamma]$ then we say that γ passes from b.

For example, if $M = R^n$ then the geodesics are straight lines and asymptotic lines are parallel. Thus a point at infinity corresponds to all lines parallel to a fixed line. If M is the hyperbolic space, which we denote it by H, there are several important models of H: the Klein model, the hyperboloid model, the Poincar ball model, the Poincar half space model, and the Belterami model. We use here the Belterami model. In the Belterami model, H is the points of the open disc:

$$H = \{(x, y) \in R^2 : \quad x^2 + y^2 < 1\}.$$

the metric tensor is given by

$$ds^2 = \frac{dx^2 + dy^2}{1 - (x^2 + y^2)} + \frac{(xdx + ydy)^2}{(1 - (x^2 + y^2))^2}. \tag{2.1}$$

Geodesics are represented by the chords, straight line segments with ideal endpoints on the boundary sphere. The infinity of H, $H(\infty)$ is interpreted as the points of the boundary sphere.

Definition 2.1. *A measurable subset K of a Hadamard manifold M is called a Kakeya set if its measure is zero and it contains a collection of geodesics with the property that for each point $b \in M(\infty)$ there exists a geodesic in K passing from b.*

Theorem 2.2. *There is a Kakeya set in H.*

Proof. We consider the Belterami model of H. By Besicovitch answer to the original Kakeya problem on R^2, one could rotate a needle using an arbitrary small area. So, We can consider a line segment L inside H, and rotate it continuously inside H with the property that the union of the all line segments achieved from rotations of L is contained in a set of R^2-measure zero. Denote by $\Gamma = \{\lambda :\}$ the set of the mentioned line segments. Each line segment λ in Γ is a portion of a chord which we denote it by λ'. Put $\Gamma' = \{\lambda' : \lambda \in \Gamma\}$. Since the union of the points of all line segments λ in Γ is contained in a set of R^2-measure zero, it is not hard to show that the union of the points of the members of Γ' has also R^2-measure zero, and from the fact that the members of Γ' are rotations of a line segment in all directions, then the union of the endpoints of the members of Γ' covers all points in the boundary of H, that is $H(\infty)$. The members of Γ' are in fact geodesics of H. So we have a Kakeya set in H if we show that the union of the members of Γ' is contained in a set of H-measure zero.

For a measurable subset W of H denote by $\mu_{_E}(W)$ and $\mu_{_H}(W)$ the R^2-measure and H-measure of W. We show that $\mu_{_E}(W) = 0$ implies $\mu_{_H}(W) = 0$. Consider the following subsets of H.

$$B_n = \{(x,y) \in R^2 : \quad x^2 + y^2 < 1 - \frac{1}{n}\}.$$

Then,

$$\mu_{_H}(W) = \int\int_W \sqrt{detA}dxdy, \tag{2.2}$$

where A is the following matrix

$$A = [a_{ij}], \quad a_{ij} = \langle e_i, e_j \rangle_{_H}, \quad e_1 = (1,0), \quad e_2 = (0,1).$$

By using of the metric tensor of H given by (2.1) we can compute

$$\begin{aligned} a_{11} &= \frac{1}{1-(x^2+y^2)} + \frac{x^2}{(1-(x^2+y^2))^2}, \\ a_{12} = a_{21} &= \frac{(x+y)^2}{(1-(x^2+y^2))^2}, \\ a_{22} &= \frac{1}{1-(x^2+y^2)} + \frac{y^2}{(1-(x^2+y^2))^2}. \end{aligned}$$

Put $W_n = W \bigcap B_n$. Since $W_n \subset B_n$ then for all $(x,y) \in W_n$, $x^2 + y^2 < \frac{1}{n}$, so

$$\begin{aligned} a_{11} &< (1+nx^2)n < (1+n)n, \\ a_{22} &< (1+ny^2)n < (1+n)n, \\ a_{12} = a_{21} &< (x+y)^2 n^2 < 4n^2. \end{aligned}$$

Then $det(A) \leq 16n^6(1+n)^2$, so by (2.2),

$$\mu_H(W_n) \leq \int\int_{W_n} \sqrt{16n^6(1+n)^2}dxdy = 4n^3(1+n)\mu_{_E}(W_n). \tag{2.3}$$

Since $\mu_{_E}(W_n) \leq \mu_{_E}(W) = 0$, then by (2.3), $\mu_{_H}(W_n) = 0$. Now, we get from

$W = \bigcup_n W_n$ that $\mu_{_H}(W) = 0$. □

It seems that the assertions are true in general for Riemannian manifolds of nonpositive curvature. Thus, the following problem is interesting to think about.

Problem : Show that there is a Kakeya set in each Riemannian Manifold of nonpositive curvature.

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ANTI-KÄHLER GEOMETRY ON COMPLEX LIE GROUPS

PARVANEH SADEGNAVEHSI

Abstract. Let G be a Lie group of even dimension and let (g, J) be a left invariant anti-Kähler structure on G. In this dissertation we study anti-Kähler structures considering the distinguished cases where the complex structure J is abelian or bi-invariant. We find that if G admits a left invariant anti-Kähler structure (g, J) where J is abelian then the Lie algebra of G is unimodular and (G, g) is a flat pseudo-Riemannian manifold. For the second case, we see that for any left invariant metric g for which J is an anti-isometry we obtain that the triple (G, g, J) is an anti-Kähler manifold.

Key words and phrases: Anti-Hermitian geometry; Norden metrics; Anti-Kähler manifold; Lie groups.

1. Introduction

Anti-Hermitian geometry can be considered as a counterpart of Hermitian geometry: an almost anti-Hermitian manifold is a triple (M, g, J), where (M, g) is a pseudo- Riemannian manifold and J is an almost complex structure on M such that J is symmetric for g. In the literature, other names are also used for this class of manifolds: Norden Manifolds [1] or almost complex manifolds with a Norden metric [2], in honour to the Russian mathematician Aleksandr P. Norden. This work is intended as an attempt to study anti-Kähler geometry on Lie groups and to motivate new properties of anti-Kähler manifolds. In this paper, we focus on anti-Kähler structures on Lie groups in the left invariant setting. In the complex geometry of Lie groups, we have two distinguished classes of left invariant complex structures, namely, abelian and bi-invariant complex structures.We study anti-Kahler structures with complex structures in each class.

Definition 1.1. *(Almost anti-Hermitian Manifold) An almost anti-Hermitian manifold is a triple (M, g, J), where M is a differentiable manifold of real dimension $2n$, J is an almost complex structure on M and g is an anti-Hermitian metric on (M, J), that is*

$$g(JX, JY) = -g(X, Y), \quad \forall X, Y \in \mathfrak{X}(M), \tag{1.1}$$

or equivalently, J is symmetric with respect to g.

If additionally J is integrable, then the triple (M, g, J) is called an anti-Hermitian manifold or complex Norden manifold.

Definition 1.2. *An Anti-Kähler manifold is an almost anti- Hermitian manifold (M, g, J) such that J is parallel with respect to the Levi-Civita connection of the pseudo-Riemannian manifold (M, g).*

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Let (M, g, J) be an almost anti-Hermitian manifold. From now on, let us denote by ∇ the Levi-Civita connection of (M, g) and we denote by $(\nabla_x J)$ the covariant derivative of J in the direction of the vector field X. We recall that

$$(\nabla_x J)Y = \nabla_x JY - J\nabla_x Y.$$

Proposition 1.3 ([3])**.** *Let (M, g, J) be an almost anti-Hermitian manifold. Then, (M, g, J) is an anti-Kähler manifold if and only if*

$$(\nabla_{JX} J)Y = \varepsilon J(\nabla_X J)Y, \quad \forall X, Y \in \mathfrak{X}(M), \tag{1.2}$$

where ε is a real constant.

2. Left invariant geometric structures on Lie groups

Definition 2.1. *A left invariant almost complex structure J on a Lie group G is called abelian when it satisfies*

$$[JX, JY] = [X, Y], \quad \forall X, Y \in \mathfrak{g}. \tag{2.1}$$

Definition 2.2. *A left invariant almost complex structure J on a Lie group is called bi-invariant if it satisfies*

$$[JX, Y] = J[X, Y](= [X, JY]), \quad \forall X, Y \in \mathfrak{g}. \tag{2.2}$$

Proposition 2.3. *Let (g, J) be a left invariant anti-Kähler structure on a Lie group G such that J is an abelian complex structure. Then (G, g, J) satisfies the condition (2.1), i.e.*

$$\nabla_{JX} Y = -J\nabla_X Y, \quad \forall X, Y \in \mathfrak{g}. \tag{2.3}$$

3. Anti-Kähler geometry on complex Lie groups

Proposition 3.1. *Let (g, J) be a left invariant almost anti-Hermitian structure on a anti-Kähler Lie group G where J is a bi-invariant complex structure on G. Then (G, g, J) is an manifold.*

4. Anti-Kähler geometry and abelian complex structures

Theorem 4.1. *Let (g, J) be a left invariant anti-Kähler structure on a Lie group G such that J is an abelian complex structure. Then (G, g) is a flat pseudo-Riemannian Lie group.*

Proof. It is sufficient to prove that $R(X, Y)Z = 0$ for all X, Y, Z in g. By the above proposition, we have $R(X, Y)Z = \nabla[X, Y]Z$ and therefore

$$R(JX, JY)Z = \nabla[JX, JY]Z = R(X, Y)Z.$$

While on the other hand, by virtue of the symmetry by pairs of the Riemannian curvature tensor of (G, g) and $(\nabla_J) \equiv 0$, $R(JX, JY)Z = -R(X, Y)Z$ and the proof is completed. □

Corollary 4.2. *Let (g, J) be a left invariant anti-Kähler structure on a Lie group G such that J is an abelian complex structure, then for all X, Y, Z in g*

$$[J[X, Y], Z] = J[[X, Y], Z]. \tag{4.1}$$

Proof.

$$
\begin{aligned}
[J[X,Y],Z] &= \nabla_{J[X,Y]}Z - \nabla_Z J[X,Y] \\
&= \frac{1}{2}\Big([J[X,Y],Z] - J[J[X,Y],JZ]\Big) + \frac{1}{2}\Big([Z,J[X,Y]] - J[Z,J^2[X,Y]]\Big) \\
&= \frac{1}{2}\Big([J[X,Y],Z] - \frac{1}{2}J[[X,Y],Z]\Big) + \frac{1}{2}\Big([Z,J[X,Y]] + \frac{1}{2}J[Z,[X,Y]]\Big) \\
&= \frac{1}{2} \times 2J\big[[X,Y],Z\big] \\
&= J\big[[X,Y],Z\big].
\end{aligned}
$$

□

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KILLING VECTOR FIELDS ON PSEUDO-RIEMANNIAN MANIFOLDS

REZA MIRZAIE

Abstract. We show that the existence of a sufficient number of Killing vector fields on a compact and connected pseudo-Riemannian manifold yields to the geodesic completeness of the manifold. We prove that in a compact pseudo-Riemannian manifold M^n_ν with positive timelike curvature and $\nu < \frac{n}{2}$, each timelike killing vector field vanishes at a point.

Key words and phrases: Pseudo-Riemannian manifold; Killing vector field.

1. Introduction

A smooth vector field X on a pseudo-Riemannian manifold M is said to be a Killing vector field if its flow consists of isometries. The existence of a nontrivial Killing vector field on M restricts its topology and geometry. An interesting problem which leads to the study of Killing vector fields is Poincare conjecture stating that an arbitrary compact connected simply connected metrizable topological 3-manifold M with the second countability axiom is homeomorphic to the 3-sphere S^3. It is proved that the Poincare conjecture is true if and only if every compact connected simply connected three-dimensional smooth manifold M admits a smooth Riemannian metric with a regular Killing vector field of constant length [3]. Motivated from many geometric subjects such Poincare conjecture and many other problems in geometry and physics, study of the Killing vector fields has been one of the active research areas in differential geometry. One can find many interesting results about Killing vector fields on Riemannian manifolds in the literature. In the pseudo-Riemannian manifolds beside the problems which have Riemannian corresponds, many new problems arise. In fact, the causal character of the Killing vector fields play important roles. For example, a compact Riemannian manifold is (geodesically) complete. But, contrary to the Riemannian case, a compact pseudo-Riemannian manifold may be geodesically incomplete. This striking fact motivated the search of sufficient assumptions under which compactness implies (geodesically) completeness of such a manifolds. Kamishima proved that a compact Lorentzian manifold which admits a timelike Killing vector field and has constant sectional curvature must be complete [9]. In a similar work, Romero and Sanchez generalized the theorem and proved that a compact Lorentzian manifold which admits a timelike Killing vector field must be complete [14] (in fact they proved the theorem in a more general case where the vector field is conformal). In the present article, we prove a similar result for pseudo-Riemannian manifolds with arbitrary index. Also, we consider the isolated fixed points of the isometries, which their infinitesimal version is the points where a Killing vector field vanishes. Motivated from

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the similar results in Lorentzian manifolds (see [2]), we show that in odd dimensional pseudo-Riemannian manifolds, a Killing vector field has no isolated vanishing point and if the dimension is even, in any neighborhood of an isolated zero of an arbitrary Killing field, the field must become spacelike, null and timelike. It is proved by Bochner [7] that in the compact Riemannian manifolds if the manifold has negative sectional curvature, then there is no non-trivial Killing vector field. The pseudo-Riemannian correspond to the negative curvature in the Riemannian manifolds is positive timelike curvature in the pseudo-Riemannian manifolds. As a result similar to Bochner's theorem, we show that in a compact pseudo-Riemannian manifold with positive timelike curvature, each Killing vector field vanishes at some point.

2. Main results

Fix throughout this article a (connected) pseudo-Riemannian manifold (M^n_ν, g) and its Levi-Civita connection ∇. A vector field on M is called a Killing vector field if for all vector fields Y and Z, $g(\nabla_X Y, Z) + g(\nabla_Z X, Y) = 0$. It is proved in [2] that if a Killing vector field X on a Lorentzian manifold M^n_1 has an isolated zero at a point p. Then, n is even and in each neighborhood of p there are points at which X is timelike, spacelike and null. We show (in Theorem 2.4) that a similar result is true in general for pseudo-Riemannian manifolds (the ideas of [2] works).

Remark 2.1. *Consider the signature matrix of $\mathbb{R}^n_\nu$, ϵ, the diagonal matrix whose diagonal entries are $\epsilon_{11} = \cdots = \epsilon_{\nu\nu} = -1$ and $\epsilon_{(\nu+1)(\nu+1)} = \cdots = \epsilon_{nn} = 1$. Each Killing vector field X on R^n_υ is in the form $a + S^*$ where, S is a skew adjoint operator on R^n_υ $(X(p) = a_p + S(p))$ (see [12] p 253). If $X(p) = 0$ for some point p, then $a = 0$ and $X(p) = S(p)$ for all $p \in R^n_\upsilon$. S is skew adjoint and its matrix in the standard basis of $\mathbb{R}^n_\nu$ obeys the equality ${}^tS = -\epsilon S \epsilon$ (see [12] page 235 Lemma 3). Thus, $detS = (-1)^n detS(det\epsilon)$. Then $detS = (-1)^n detS$, which implies that for odd numbers n, S is singular and we have zero eigenvalues. Thus, there is a line L such that $X(q) = 0$ for all $q \in L$.*

If in Remark 2.1, n is even, we have the following argument.

Remark 2.2. *Suppose that $X(p) = 0$ and let A be a non-degenerate (affine) subspace of $\mathbb{R}^n_\nu$ containing p and $\dim A = n-1$. Put $Y = tan_A X$, where tan_A denotes the normal projection map on A. Y is a killing vector field along A. Since $n-1$ is odd then by the above remark, there is a line L on A passing from p such that Y is zero along L. Then X is normal to A along L. Let W be the unite normal vector field on A. Parametrize L as a geodesic $\gamma(s)$, $s \in I$. We have along L, $X = f(s)W$, where f is a differentiable function on I. Since X is a Jaccobi field along γ, then $X'' = R_{X\gamma'}\gamma'$. Thus $f''(s)W = f(s)R_{w\gamma'}\gamma'$. Since the curvature is zero then $g(R_{W\gamma'}\gamma', W) = 0$. Thus $f''(s) = 0$ and along L, $X = csW$ for a constant number c.*

We get from the above remarks the following theorem.

Theorem 2.3. *Let X be a killing vector field on R^n_υ and $X(p) = 0$ for some point p.*
(1) *If n is odd, then X is zero along a line L passing from p.*
(2) *If n is even, then for all affine subspaces S containing p, X is normal to S along a line contained in S and passing from p.*

Theorem 2.4. *If p is an isolated zero point of the Killing vector field X on a connected pseudo-Riemannian manifold M^n_ν, then n is even and in each neighborhood of p there are points where X becomes timelike, spacelike and null.*

Proof. Without lose of generality, we can assume that $M = R^n_\nu$(for each point $p \in M$ use the exponential map $exp : T_pM \to M$ to move the problem from M to T_pM which is equivalent to R^n_ν). By the above theorem (part 1), it is clear that n must be even. In part 2 of the above theorem, choose S an affine non-degenerate subspace with index $v-1$. Then, the vector field W must be timelike and X will be timelike along a line in S containing p. If we choose S with the index ν, then W will be spacelike and similarly, by suitable choose of S, X will be null along a line containing p. □

Theorem 2.5. *If M^n_ν is a compact pseudo-Riemannian manifold with ν linearly independent timelike killing vector fields, then M is geodesically complete.*

Proof. We prove the theorem in the following two steps.
Step 1. Let $K_1, \dots, K_\nu$ be linearly independent timelike vectors in $\mathbb{R}^n_\nu$ and $c_1, \dots, c_\nu$ be constant numbers. Let $\Omega = \{v \in R^n_\nu :< v, v >= -1, \ \ < v, K_i >= c_i, 1 \le i \le \nu\}$. Consider the linear subspace S of $\mathbb{R}^n_\nu$ generated by $K_1, \dots, K_\nu$ and let $e_1, \dots, e_\nu$ be an orthonormal basis for S. Then, for some constant numbers a_{ij}, $e_i = \sum_j a_{ij}K_j$. Consequently, $< v, e_i >= \sum_j a_{ij}c_j$. Thus, $< v, e_i >$ is constant which we denote it by d_i. Consider an orthonormal basis $\{e_{\nu+1}, \dots, e_n\}$ for $S^\perp$. Then $\{e_1, \dots, e_n\}$ is an orthonormal basis for $\mathbb{R}^n_\nu$ ($e_1, \dots, e_\nu$ timelike, $e_{\nu+1}, \dots, e_n$ spacelike).
If $v = \sum_i v_i e_i \in \Omega$, then

$$\Omega = \{v = (d_1, \dots, d_\nu, v_{\nu+1}, \dots, v_n) : \sum_{i=\nu+1}^{n} v_i^2 = -1 + \sum_{i=1}^{\nu} d_i^2\}.$$

$-1 + \sum_{i=1}^{\nu} d_i^2$ is a (positive) constant number which we denote it by c. This means that Ω is homeomorphic to the standard sphere $\{(v_{\nu+1}, \dots, v_n) : \sum_{i=\nu+1}^{n} v_i^2 = c\}$ of $\mathbb{R}^{n-\nu}$. Therefore, Ω is compact.
Step 2. Let $c_1, \dots, c_\nu$ be constant numbers and put

$$\Omega_p = \{v \in T_pM : \ < v, v >= -1, \ \ < v, K_i >= c_i\}.$$

By Step 1, Ω_p is a compact subset of T_pM.
Let $\gamma : I \to M$ be a unite speed timelike geodesic in M. Since K_i is Killing vector field, $< K_i, \gamma'(t) >$ is constant along γ which we denote it by c_i. Put $\bar{\Omega} = \bigcup_{p \in M} \Omega_p$. Since M is compact, then $\bar{\Omega}$ is compact. We have $\{(\gamma(t), \gamma'(t)) : t \in I\} \subset \bar{\Omega}$. Therefore, γ is complete. A similar proof works if γ is spacelike or null. □

Theorem 2.6. *If M^n_ν is a compact pseudo-Riemannian manifold with positive timelike curvature and $\nu < \frac{n}{2}$, then each timelike Killing vector field must be zero at least in one point of M.*

Proof. Let X be a Killing vector field on M which is nonzero at all points of M. We will get a contradiction as follows:
Put $f : M \to R$, $f(x) =< X(x), X(x) >$. Since M is compact, f has a maximum point p. Since p is a critical point of f, then for all $v \in T_pM$, $v[f] = 0$ which implies $< \nabla_v X, X >= 0$. Thus $v \in X(p)^\perp$. We need the following claim.
Claim: There is a spacelike vector $v \in X(p)^\perp$ such that $\nabla_v X$ is nonspaclike at p. Proof of the claim: Suppose that for all $v \in T_pM$, $\nabla_v X$ is timelike. Without lose

of generality we can suppose that $T_pM = R^n_\nu$ and $X \in R^\nu_\nu$.
Put $S : T_pM(= R^{n-\nu}) \to R^\nu,\ \ S(v) = \nabla_v X$. We have $dim(kerS) \geq (n-\nu)-\nu > 0$. Then, there is a non-zero spacelike vector v such that $v \in ker(S)$ and $\nabla_v X = 0$. Now, consider the following computations

$$\nabla_v \nabla_v f = 2\nabla_v < \nabla_v X, X >= 2 < \nabla_v \nabla_v X, X > +2 < \nabla_v X, \nabla_v X > . \tag{2.1}$$

Since X is Killing, then

$$< \nabla_v \nabla_v X = R(v, X)v. \tag{2.2}$$

So, we get from (2.1) and (2.2) that

$$\begin{aligned} \nabla_v \nabla_v f &= 2 < R(v, X)v, v > +2 < \nabla_v X, \nabla_v X > \\ &= -\kappa(v, X)(< v, v >< X, X >) + 2 < \nabla_v X, \nabla_v X > . \end{aligned} \tag{2.3}$$

By the above claim, there is a spacelike vector v orthogonal to X at p such that $< \nabla_v X, \nabla_v X >$ is nonnegative. Thus, by our assumption on the curvature, (2.3) yields to $\nabla_v \nabla_v f > 0$. Let α be a geodesic with initial velocity v, then

$$\nabla_v \nabla_v f = \frac{d}{dt}(f \circ \alpha(t))_{|t=0}.$$

This means that f has local minimum at p, which is in contrast with the assumption that p is a maximum point for f. $\square$

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GENERALIZED η-RICCI SOLITONS ON KENMOTSU MANIFOLDS ASSOCIATED TO THE QUARTER SYMMETRIC NON-METRIC ϕ-CONNECTION

SHAHROUD AZAMI AND SAKINEH HAJIAGHASI

ABSTRACT. In this paper, we investigate Kenmotsu manifolds admitting generalized η-Ricci solitons associated to the quarter symmetric non-metric ϕ-connection. We provide two examples of generalized η-Ricci solitons on a Kenmotsu manifolds to prove our results.

Key words and phrases: Kenmotsu manifolds; generalized η-Ricci soliton; quarter symmetric non-metric ϕ-connection.

1. Introduction

The Kenmotsu manifold was introduced by Kenmotsu [8] in 1972 as a new class of almost contact metric manifolds. Kenmotsu manifolds are very closely related to the warped product manifolds. In dimensional three, Kenmotsu manifold is locally modeled on the product of a circle and a hyperbolic plane. The Kenmotsu manifold is characterized by its contact structure, which is a special type of geometric structure that arises in the study of certain physical systems.

In 1982, Hamilton [7] introduced the notion of Ricci soliton as a generalization of Einstein metrics and a special solution to Ricci flow on a Riemannian manifold. A Ricci soliton [6] is a triplet (g, V, λ) on a pseudo-Riemannian manifold M such that $\mathcal{L}_V g + 2S + 2\lambda g = 0$, where $\mathcal{L}_V$ is the Lie derivative in direction the potential vector field V, S is the Ricci tensor, and λ is a real constant.

Motivated by the above studies M. D. Siddiqi [10] introduced the notion of generalized η-Ricci soliton as follows

$$\mathcal{L}_V g + 2\mu V^\flat \otimes V^\flat + 2S + 2\lambda g + 2\rho\eta \otimes \eta = 0. \tag{1.1}$$

Motivated by [2, 3, 9] and the above works, we study generalized η-Ricci solitons on Kenmotsu manifolds associated the quarter symmetric non-metric ϕ-connection. We give an example of generalized η-Ricci soliton on a Kenmotsu manifold associated the quarter symmetric non-metric ϕ-connection.

2. Preliminaries

A n-dimensional metric manifold (M, g) is said to be a almost contact manifold [1], with an almost contact structure (ϕ, ξ, η, g), if there exist a $(1,1)$-tensor field ϕ, a vector field ξ and a 1-form η such that

$$\phi^2(X) = -X + \eta(X)\xi, \eta(\xi) = 1, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

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for all vector fields X, Y on M. In this case, we have $\phi\xi = 0$, $\eta \circ \phi = 0$, and $\eta(X) = g(X, \xi)$. A almost contact manifold M is called Kenmotsu manifold [5], if

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{2.3}$$

for all vector fields X, Y on M. In a Kenmotsu manifold, we have

$$\nabla_X \xi = X - \eta(X)\xi, \tag{2.4}$$

$$(\nabla_X \eta)Y = g(X, Y) - \eta(X)\eta(Y), \tag{2.5}$$

where ∇ is the Levi-Civita connection with respect to the metric g. Using (2.4) and (2.5), we find

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \quad R(X, \xi)Y = g(X, Y)\xi - \eta(Y)X,$$

for all vector fields X, Y, Z, where R is the Riemannian curvature tensor. The Ricci tensor S of a Kenmotsu manifold M is defined by $S(X, Y) = \sum_{i=1}^{n} g(R(e_i, X)Y, e_i)$ and we have $S(X, \xi) = -(n-1)\eta(X)$, for all vector field X on M.

Let M be a Kenmotsu manifold and ∇ be the Levi-Civita connection on M. Then a quarter symmetric non-metric ϕ-connection $\bar{\nabla}$ [4, 5] on M with respect to Levi-Civita connection ∇ is defined by

$$\bar{\nabla}_X Y = \nabla_X Y - \eta(X)\phi Y + g(X, Y)\xi - \eta(Y)X - \eta(X)Y, \tag{2.6}$$

for all vector fields X, Y on M. Let $\bar{R}$ and $\bar{S}$ be the curvature tensors and the Ricci tensors of the connection $\bar{\nabla}$, respectively, that is

$$\bar{R}(X, Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]} Z, \quad \bar{S}(X, Y) = \sum_{i=1}^{n} g(\bar{R}(e_i, X)Y, e_i).$$

On Kenmotsu manifolds, applying (2.6) and the above relation we find

$$\bar{R}(X, Y)Z = R(X, Y)Z + g(Y, Z)X - g(X, Z)Y + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y, \tag{2.7}$$

and

$$\bar{S}(X, Y) = S(X, Y) + (n-1)g(X, Y) + (n-1)\eta(X)\eta(Y), \tag{2.8}$$

for all vector fields X, Y, Z on M, where S denotes the Ricci tensor of the connection ∇. Let r and $\bar{r}$ be the scalar curvature of the Levi-Civita connection ∇ and the quarter symmetic non-metric ϕ-connection $\bar{\nabla}$. The equation (2.8) implies that

$$\bar{r} = r + n^2 - 1. \tag{2.9}$$

The generalized η-Ricci soliton associated to the quarter symmetric non-metric ϕ-connection is defined by

$$\alpha\bar{S} + \frac{\beta}{2}\overline{\mathcal{L}}_V g + \mu V^\flat \otimes V^\flat + \rho\eta \otimes \eta + \lambda g = 0, \tag{2.10}$$

where $\bar{S}$ denotes the Ricci tensor of the connection $\bar{\nabla}$, $(\overline{\mathcal{L}}_V g)(Y, Z) := g(\tilde{\nabla}_Y V, Z) + g(Y, \tilde{\nabla}_Z V)$, $V^\flat$ is the canonical 1-form associated to V that is $V^\flat(X) = g(V, X)$ for all vector field X, λ is a smooth function on M, and α, β, μ, ρ are real constant such that $(\alpha, \beta, \mu) \neq (0, 0, 0)$. Note that

$$(\overline{\mathcal{L}}_V g)(X, Y) = \mathcal{L}_V g(X, Y) - 2\eta(V)g(X, Y) - g(X, \phi V)\eta(Y) - g(Y, \phi V)\eta(X).$$

3. Main results

A Kenmotsu manifold is said to η-Einstein with respect to the quarter symmetric non-metric ϕ-connection if its Ricci tensor $\bar{S}$ is of the form $\bar{S} = ag + b\eta \otimes \eta$, where a and b are smooth functions on manifold. Let M be a Kenmotsu manifold. Now, we consider M satisfies the generalized η-Ricci soliton (2.10) associated to the quarter symmetric non-metric ϕ-connection and the potential vector field V is a pointwise collinear vector field with the structure vector field ξ, that is, $V = f\xi$ for some function f on M. Using (2.6) we get

$$\bar{\mathcal{L}}_{f\xi} g(X,Y) = (Xf)\eta(Y) + (Yf)\eta(X) - 2f\eta(X)\eta(Y)),$$

for all vector fields X, Y on M. Also, we have

$$\alpha \bar{S}(X,Y) = -\lambda g(X,Y) + ((n-1)\alpha + \lambda)\eta(X)\eta(Y), \tag{3.1}$$

which implies $\alpha\bar{r} = \lambda(1-n) + (n-1)\alpha$. Therefore, this leads to the following theorem.

Theorem 3.1. *Let (M, g, ϕ, ξ, η) be a Kenmotsu manifold. If M admits a generalized η-Ricci soliton $(g, V, \alpha, \beta, \mu, \rho, \lambda)$ with respect to the quarter symmetric non-metric ϕ-connection such that $\alpha \neq 0$ and $V = f\xi$ for some smooth function f on M, then M is an η-Einstein manifold with respect to the quarter symmetric non-metric ϕ-connection.*

Now, let M be an η-Einstein Kenmotsu manifold with respect to the quarter symmetric non-metric ϕ-connection and $V = \xi$. Then we get $\bar{S} = ag + b\eta \otimes \eta$ for some functions a and b on M. Hence, we can state the following theorem.

Theorem 3.2. *Suppose that M is a η-Einstein Kenmotsu manifold with respect to the quarter symmetric non-metric ϕ-connection such that $\bar{S} = ag + b\eta \otimes \eta$ for some function a and constant b on M. Then manifold M satisfies a generalized η-Ricci soliton $(g, \xi, \alpha, \beta, \mu, -b\alpha - \mu + \beta, -a\alpha)$ with respect to the quarter symmetric non-metric ϕ-connection.*

Now assume that a Kenmotsu manifold with respect to the quarter symmetric non-metric ϕ-connection satisfying the condition $\bar{R}(X,Y).\bar{S} = 0$ (or $\bar{S}.\bar{R} = 0$) for all vector fields X, Y on M. Then from [4], we have $\bar{S}(X,Y) = (n-1)\eta(X)\eta(Y)$. Thus we have the following theorem.

Theorem 3.3. *Let M be a Kenmotsu manifold with the quarter symmetric non-metric ϕ-connection satisfy the condition $\bar{R}.\bar{S} = 0$ or $\bar{S}.\bar{R} = 0$. Then manifold M satisfies a generalized η-Ricci soliton $(g, \xi, \alpha, \beta, \mu, -(n-1)\alpha - \mu + \beta, 0)$ with respect to the quarter symmetric non-metric ϕ-connection.*

Definition 3.4. *Let M be a Kenmotsu manifold with the quarter symmetric non-metric ϕ-connection $\bar{\nabla}$. The projective curvature tensor $\bar{P}$ with respect to the quarter symmetric non-metric ϕ-connection on M is defined by*

$$\bar{P}(X,Y)Z = \bar{R}(X,Y)Z - \frac{1}{n-1}\left(\bar{S}(Y,Z)X - \bar{S}(X,Z)Y\right). \tag{3.2}$$

We have the following theorem.

Theorem 3.5. *Let M be a ϕ-projectively flat (or quasi-projectively flat or $\bar{P}.\bar{S} = 0$) Kenmotsu manifold with respect to the quarter symmetric non-metric ϕ-connection. Then manifold M satisfies a generalized η-Ricci soliton $(g, \xi, \alpha, \beta, \mu, -(n-1)\alpha - \mu + \beta, 0)$, with respect to the quarter symmetric non-metric ϕ-connection.*

Definition 3.6. *A vector field V is said to a conformal Killing vector field if*

$$(\mathcal{L}_V g)(X,Y) = 2hg(X,Y), \tag{3.3}$$

for all vector fields X, Y, where h is some function on M.

Theorem 3.7. *If the metric g of a Kenmotsu manifold satisfies the generalized η-Ricci soliton $(g, V, \alpha, \beta, \mu, \rho, \lambda)$ where V is and conformally Killing vector field with respect to the quarter symmetric non-metric ϕ-connection, that is $\overline{\mathcal{L}}_V g = 2hg$ then $((n-1)\alpha + \beta h + \rho + \lambda)\xi + \mu\eta(V)V = 0$.*

Definition 3.8. *A nonvanishing vector field V on pseudo-Riemannian manifold (M, g) is called torse-forming* [11]*, if*

$$\nabla_X V = fX + \omega(X)V, \tag{3.4}$$

for all vector field X, where ∇ is the Levi-Civita connection of g, f is a smooth function and ω is a 1-form.

Theorem 3.9. *If the metric g of a Kenmotsu manifold satisfies the generalized η-Ricci soliton $(g, V, \alpha, \beta, \mu, \rho, \lambda)$ with respect to quarter symmetric non-metric ϕ-connection where V is the torse-forming vector filed and satisfied in* (3.4)*, then*

$$\lambda = -\frac{1}{n}\left[\alpha(r + n^2 - 1) + \rho + \beta\omega(V) + -n\beta\eta(V) + \mu|V|^2\right] - \beta f.$$

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LAPLACIAN COMPARISON ON RICCI SOLITONS

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ABSTRACT. We consider a condition on the weighted Ricci curvature involving vector fields, under this condition we prove Laplacian comparison theorem on the metric measure space M^n.

Key words and phrases: Ricci curvature; metric measure space.

1. Introduction

Let (M,g) be a complete n-dimensional Riemannian manifold and $d\mu := e^{-\phi}dV$ with ϕ a fixed smooth real-valued function on M. The triple $(M,g,d\mu)$ is called a smooth metric measure space carries a natural analog of the Ricci curvature, the so-called m-Bakry-Émery Ricci curvature, which is defined as follows:

$$\mathrm{Ric}_\phi^m := \mathrm{Ric} + \mathrm{Hess}\phi - \frac{\nabla\phi\otimes\nabla\phi}{m-n}, \quad (n < m \leq \infty).$$

In particular, when $m=\infty$, $\mathrm{Ric}_\phi^\infty := \mathrm{Ric}_\phi = \mathrm{Ric} + \mathrm{Hess}\phi$ is the classical Bakry-Émery Ricci curvature introduced in [1]. There is also a natural analog of the Laplacian, called the weighted Laplacian, denoted by $\Delta_\phi = \Delta - \nabla\phi.\nabla$, which is a self-adjoint operator in $L^2(M,d\mu)$. A useful fact is that Δ_ϕ relates to the Bakry-Émery curvature via the following weighted Bochner formula, see [3, 4].

$$\frac{1}{2}\Delta_\phi|\nabla u|^2 = |\nabla^2 u|^2 + \nabla u.\nabla\Delta_\phi u + \mathrm{Ric}_\phi(\nabla u,\nabla u). \tag{1.1}$$

Laplacian comparison theorem was established in [5] under a Ricci curvature condition that is modified as follows

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_V g - \frac{1}{N-n}V\otimes V \geq -\lambda g,$$

where V is a vector field, n is the dimension of the manifold M, and N is a number strictly greater than n.
The weighted Laplacian comparison theorem with m-Bakry-Émery curvature studied in [2]. After that Wei and Wylie in [7] proved the weighted Laplacian comparison theorem on a smooth metric measure space with bounded ∞-Bakry-Émery. They also derived the weighted Myer's Theorem when

$$\mathrm{Ric}_\phi \geq (n-1)K > 0.$$

Recently, authors in [6] studied Volume comparison and Laplacian comparison theorems considering a condition on the Ricci curvature involving vector fields. Their basic assumption on the Ricci curvature tensor was as follows

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_V g \geq -\lambda g. \tag{1.2}$$

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Speaker: Sakineh Hajiaghasi .

Here $\lambda \geq 0$ is constant and V is a smooth vector field which satisfies the following condition

$$|V|(y) \leq \frac{K}{d(y,O)^{\alpha}}, \tag{1.3}$$

for any $y \in M$. Here $d(y,O)$ represents the distance from a fixed base point O to y, and $K \geq 0$ and $0 \leq \alpha < 1$ are constants. They proved the following Laplacian comparison theorem.

Theorem 1.1. *Assume that on a Riemannian manifold (M,g), (1.2) and (1.3) hold. Let $s = d(y,x)$ be the distance from any point y to some fixed point x, and $\gamma : [0,s] \to M$ as a normal minimal geodesic with $\gamma(0) = x$ and $\gamma(s) = y$. Then in the distribution sense*

$$\Delta s - \frac{n-1}{s} \leq \frac{\lambda}{3}s + < V, \nabla s > + \frac{C(\alpha)K}{s^{\alpha}}.$$

Motivated by the above theorem, we want to study Laplacian comparison on a metric measure space M^n involving the following Ricci curvature

$$\mathrm{Ric}_{\phi} + \frac{1}{2}\mathcal{L}_V g,$$

here $\mathrm{Ric}_{\phi} = \mathrm{Ric} + \mathrm{Hess}\phi$. We state our main result as follows

Theorem 1.2. *Let $(M, g, e^{-\phi}dV)$ be a smooth metric measure space. Assume that the following conditions hold on M*

$$\mathrm{Ric}_{\phi} + \frac{1}{2}\mathcal{L}_V g \geq -\lambda g, \qquad |V|(y) \leq \frac{K}{d(y,O)^{\alpha}}.$$

Here $\lambda \geq 0$, $K \geq 0$, and $0 \leq \alpha < 1$ are constants; $d(y,O)$ represents the distance from O to y for any $y \in M$ and fixed point $O \in M$. Consider $\gamma : [0,s] \to M$ as a normal minimal geodesic joins x to y. Moreover consider the following condition on function ϕ.

$$|\phi(y) - \phi(z)| \leq K_1 d(y,z)^{\alpha}, \tag{1.4}$$

Then the following inequality holds in the sense of distribution

$$\Delta_{\phi} s - \frac{n-1}{s} \leq \frac{\lambda}{3}s + < V, \nabla s > + \frac{C(\alpha)K}{s^{\alpha}} + \frac{4K_1}{s^{1-\alpha}}.$$

Here $C(\alpha)$ denotes constant depends on α.

2. Proof of Main results

From weighted Bochner formula (1.1), for $s = d(x,y)$, we have

$$0 = |\nabla^2 s|^2 + < \nabla \Delta_{\phi} s, \nabla s > + \mathrm{Ric}_{\phi}(\nabla s, \nabla s).$$

Since $|\nabla^2 s|^2 \geq \frac{(\Delta s)^2}{n-1}$, we obtain

$$\frac{\partial}{\partial s}\Delta_{\phi} s + \frac{(\Delta s)^2}{n-1} < -\mathrm{Ric}_{\phi}(\nabla s, \nabla s).$$

So

$$\frac{1}{s^2}\frac{\partial}{\partial s}(s^2\Delta s)+\frac{1}{n-1}(\Delta s-\frac{n-1}{s})^2$$

$$\leq \frac{n-1}{s^2}-\mathrm{Ric}_\phi(\nabla s,\nabla s)+\frac{\partial}{\partial s}<\nabla s,\nabla\phi> . \tag{2.1}$$

Multi-playing both sides of (2.1) by s^2 and then integrating yields

$$\Delta s\leq\frac{n-1}{s}-\frac{1}{s^2}\int_0^s t^2\mathrm{Ric}_\phi(\nabla s,\nabla s)+\frac{1}{s^2}\int_0^s t^2\frac{\partial}{\partial s}<\nabla s,\nabla\phi> .$$

Considering orthonormal frame $\{e_i\}$, with $e_1=\gamma^{'}(t)$, we have

$$\mathrm{Ric}_\phi(\gamma^{'}(t),\gamma^{'}(t))\geq-\lambda-\frac{\partial}{\partial t}<V,\gamma^{'}(t)> .$$

Thus

$$\begin{aligned}\Delta s-\frac{n-1}{s} \quad &\leq \quad \frac{\lambda}{3}s+\frac{1}{s^2}[t^2<V,\gamma^{'}(t)>(\gamma(t))|_0^s-2\int_0^s t<V,\gamma^{'}(t)>dt]\\ &\quad -\frac{1}{s^2}[t^2<\gamma^{'}(t),\nabla\phi>(\gamma(t))|_0^s-2\int_0^s t<\gamma^{'}(t),\nabla\phi>dt].\end{aligned}$$

In both cases whenever $s\leq d_0$ or $s>d_0$, we can obtain that

$$-\frac{2}{s^2}\int_0^s t<V,\gamma^{'}(t)>dt\leq\frac{C(\alpha)K}{s^\alpha}.$$

On the other hand by (1.4), we obtain

$$\frac{2}{s^2}\int_0^s t<\gamma^{'}(t),\nabla\phi>dt\leq\frac{4K_1}{s^{1-\alpha}},$$

this completes the proof. □

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THE SIZE OF QUASICONTINUOUS MAPS ON KHALIMSKY LINE

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Abstract. In the following text we show if D is Khalimsky line (resp. Khalimsky plane, Khalimsky circle, Khalimsky sphere), then for topological space X we show the collection of all quasicontinuous maps from D to X has cardinalirty $card(X)^{\aleph_0}$.

Key words and phrases: Khamilsky circle; Khamilsky line; quasicontinuous.

1. Introduction

Quasicontinuity is one of the weaker forms of continuity. In topological spaces Y, Z:

- Z^Y denotes the collection of all maps from Y to Z,
- $Q(Y,Z)$ denotes the collection of all quasicontinuous maps from Y to Z,
- $C(Y,Z)$ denotes the collection of all continuous maps from Y to Z.

There we say $f : Y \to Z$ is quasicontinuous at $y \in Y$, if for each open neighbourhood G of y and open neighbourhood H of $f(y)$, there exists nonempty open subset W of G such that $f(W) \subseteq H$. Also we say $f : Y \to Z$ is quasicontinuous if f is quasicontinuous at each point of Y [2]. It is clear that $C(Y,Z) \subseteq Q(Y,Z) \subseteq Z^Y$.

By Khalimsky line we mean $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$ equipped with topological base $\{\{2n+1\} : n \in \mathbb{Z}\} \cup \{\{2n-1, 2n, 2n+1\} : n \in \mathbb{Z}\}$ [1]. Let's denote Khalimsky line by $\mathcal{K}$ and:

$$V(n) := \begin{cases} \{2k+1\} & n = 2k+1 \in 2\mathbb{Z}+1\,, \\ \{2k-1, 2k, 2k+1\} & n = 2k \in 2\mathbb{Z}\,, \end{cases}$$

then $V(n)$ is the smallest open neighbourhood of each $n \in \mathcal{K}$. We call $\mathcal{K}^2$, Khalimsky plane.

Let's mention $\aleph_0 = card(\mathbb{N})$ denotes the least infinite cardinal number.

In this text we compute the cardinality of $Q(\mathcal{K}, X)$.

2. Quasicontinuous maps on Khalimsky line and Khalimsky plane

In this section we show $card(Q(\mathcal{K}^n, X)) = card(X)^{\aleph_0}$ for each topological space X.

Theorem 2.1. *For topological space X, $k \in \mathbb{Z}$, and $f : \mathcal{K} \to X$,*

1. *f is quasicontinuous at $2k-1$,*

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2. *if there exists* i *such that* $f(2k) = f(2k + (-1)^i)$*, then* f *is quasicontinuous in* $2k$*,*
3. *in metric space* (X, d) *if* f *is quasi continuous at* $2k$*, then there exists* i *such that* $f(2k) = f(2k + (-1)^i)$*.*

Proof. (1) $2k - 1$ is an isolated point of $\mathcal{K}$, so any map on $\mathcal{K}$ is continuous (quasicontinuous) at $2k - 1$.

(2) Suppose there exists i such that $f(2k) = f(2k + (-1)^i)$, G is an open neighbourhood of $2k$ and H is an open neighbourhood of $f(2k)$, then

$$W := \{2k + (-1)^i\} \subseteq V(2k) \subseteq G,$$

and W is a nonempty open subset of G, moreover

$$f(W) = \{f(2k + (-1)^i)\} = \{f(2k)\} \subseteq H.$$

Thus f is quasicontinuous at $2k$.

(3) For metric space (X, d) suppose f is quasicontinuous at $2k$. For each $n \geq 1$ there exists nonempty open subset W_n of $V(2k)$ such that

$$f(W_n) \subseteq \{x \in X : d(x, f(2k)) < \frac{1}{n}\}.$$

All nonempty open subsets of $V(2k)$ are $V(2k) = \{2k-1, 2k, 2k+1\}, \{2k-1\}$, $\{2k+1\}$. Hence, $2k - 1 \in W_n$ or $2k + 1 \in W_n$. Therefore there exists $j_n \in \{-1, 1\}$ with $2k + j_n \in W_n$ and $d(f(2k), f(2k + j_n)) < \dfrac{1}{n}$. The sequence $\{2k + j_n\}_{n\geq 1}$ has at least one of the constant subsequences $\{2k + 1\}_{m\geq 1}$ or $\{2k - 1\}_{m\geq 1}$.
Suppose $\{2k + (-1)^i\}_{n\geq 1}$ is the constant subsequence of $\{2k + j_n\}_{n\geq 1}$. So $f(2k) = \lim\limits_{n\to\infty} f(2k + j_n) = \lim\limits_{m\to\infty} f(2k + (-1)^i) = f(2k + (-1)^i)$ which completes the proof. □

Theorem 2.2. *In topological space* X *we have*

$$card(Q(\mathcal{K}, X)) = card(X)^{\aleph_0} .$$

In particular for infinite countable X*,*

$$card(Q(\mathcal{K}, \mathcal{K})) = card(Q(\mathcal{K}, X)) = \aleph_0^{\aleph_0} = 2^{\aleph_0},$$

and $card(Q(\mathcal{K}, \mathbb{R})) = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0}$*.*

Proof. Suppose $\mathfrak{S} = \{x_n\}_{n\in\mathbb{Z}}$ is a bisequence in X, by Theorem 2.1, $f_{\mathfrak{S}} : \mathcal{K} \to X$ with $f_{\mathfrak{S}}(2k - 1) = f_{\mathfrak{S}}(2k) = x_k$ $(k \in \mathbb{Z})$ is quasicontinuous. Therefore

$$\begin{aligned} card(Q(\mathcal{K}, X)) &\geq card\{\mathfrak{S} : \mathfrak{S} \text{ is a bisequence in } X\} \\ &= card(X^{\mathbb{Z}}) = card(X)^{card(\mathbb{Z})} = card(X)^{\aleph_0} . \end{aligned}$$

On the other hand

$$card(X)^{\aleph_0} = card(X^{\mathcal{K}}) \overset{(X^{\mathcal{K}} \supseteq Q(\mathcal{K}, X))}{\geq} card(Q(\mathcal{K}, X)),$$

which completes the proof by Schröder-Bernstein theorem. □

Corollary 2.3. *If* X *is a totally disconnected space (e.g., Cantor set or discrete space), then* $C(\mathcal{K}, X)$ *is just the collection of constant maps, therefore* $card(X) = card(C(\mathcal{K}, X))$*. In particular for* $D \in \{\mathbb{Z}, \mathbb{N}, \mathbb{Q}\}$*, we have*

$$card(C(\mathcal{K}, D)) = card(D) = \aleph_0 < 2^{\aleph_0} = card(Q(\mathcal{K}, D)) .$$

Theorem 2.4. *For $j \in \mathbb{Z}$ let*

$$j^* := \begin{cases} j & j \in 2\mathbb{Z}+1\,, \\ \\ j-1 & j \in 2\mathbb{Z}\,, \end{cases}$$

then for each $(a_1, \ldots, a_n) \in \mathcal{K}^n$ (equipped with product topology), topological space X, and $f : \mathcal{K}^n \to X$ we have

1. *$V(a_1) \times \cdots \times V(a_n)$ is the smallest open neighbourhood of $(a_1, \ldots, a_n)$,*
2. *$\{(a_1^*, \ldots, a_n^*)\}$ is an open subset of $V(a_1) \times \cdots \times V(a_n)$,*
3. *if $f(a_1, \ldots, a_n) = f(a_1^*, \ldots, a_n^*)$, then f is quasicontinuous at $(a_1, \ldots, a_n)$,*
4. *$card(Q(\mathcal{K}^n, X)) = card(X)^{\aleph_0} (= card(X^{\ \mathcal{K}^n}))$.*

Proof. (1, 2) Use properties of product topology.

(3) Use a similar method described in Theorem 2.1.

(4) $(2\mathbb{Z}+1)^n$ is infinite countable, so we may suppose $(2\mathbb{Z}+1)^n = \{u_1, u_2, \ldots\}$ with distinct u_is. Suppose $\mathfrak{S} = \{x_i\}_{i \in \mathbb{N}}$ is an arbitrary sequence in X, by item (3), $f_{\mathfrak{S}} : \mathcal{K}^n \to X$ with $f_{\mathfrak{S}}(a_1, \ldots, a_n) = x_k$ (where $k \in \mathbb{N}$ and $(a_1^*, \ldots, a_n^*) = u_k$) is quasicontinuous. Using a similar method described in Theorem 2.2 , we have $card(Q(\mathcal{K}^n, X)) = card(X)^{\aleph_0}$. □

3. Quasicontinuous maps on Khalimsky circle and Khalimsky sphere

In topological space W suppose $\infty \notin W$ and let $A(W) := W \cup \{\infty\}$. Consider $A(W)$ with topology

$$\big\{U \subseteq W : U \text{ is an open subset of } W\big\} \cup \big\{U \subseteq A(W) : W \backslash U \text{ is a closed compact subset of } W\big\},$$

we call $A(W)$ one point compactification or Alexandroff compactification of W [3]. One point compactification of Khalimsky line is called Khalimsky circle and one point compactification of Khalimsky plane is called Khalimsky sphere. In this section we show $card(Q(A(\mathcal{K}^n), X)) = card(X)^{\aleph_0}$ for each topological space X and $n \geq 1$.

Remark 3.1. *For $n \geq 1$, compact subsets of $\mathcal{K}^n$ are finite. Suppose E is a compact subset of $\mathcal{K}^n$, thus $\{V(a_1) \times \cdots \times V(a_n) : (a_1, \ldots, a_n) \in E\}$ is an open cover of E, hence there exists finite subset G of E such that*

$$E \subseteq \bigcup \Big\{ V(a_1) \times \cdots \times V(a_n) : (a_1, \ldots, a_n) \in G \Big\},$$

since $V(a_1) \times \cdots \times V(a_n)$s and G are finite, E is finite too.

Theorem 3.2. *$card(Q(A(\mathcal{K}^n), X)) = card(X)^{\aleph_0}$ for topological space X and $n \geq 1$.*

Proof. Using the same notations as in Theorem 2.4 for each sequense $\mathfrak{S} = \{x_i\}_{i \in \mathbb{N}}$ in X, define $g_{\mathfrak{S}} : \mathcal{K}^n \to X$ with

$$g_{\mathfrak{S}}(a) := \begin{cases} x_k & a = (a_1, \cdots, a_n) \in \mathcal{K}^n, \quad (a_1^*, \cdots, a_n^*) = u_k, a_1^* > 0\,, \\ \\ x_1 & a = (a_1, \cdots, a_n) \in \mathcal{K}^n, \quad a_1^* < 0\,, \\ \\ x_1 & a = \infty\,, \end{cases}$$

then for $a \in A(\mathcal{K}^n)$, we have the following cases.

- $a = (a_1, \dots, a_n) \in \mathcal{K}^n$: in this case for each open neighbourhood U of a and open neighbourhood V of $g_{\mathfrak{S}}(a)$, $V(a_1) \times \cdots \times V(a_n)$ is the smallest open neighbourhood of a and $W := \{(a_1^*, \dots, a_n^*)\} (\subseteq V(a_1) \times \cdots \times V(a_n) \subseteq U)$ is a nonempty open subset of U, also
$$g_{\mathfrak{S}}(W) = \{g_{\mathfrak{S}}(a_1^*, \dots, a_n^*)\} = \{g_{\mathfrak{S}}(a_1, \dots, a_n)\} \subseteq V \ ,$$
therefore in this case $g_{\mathfrak{S}}$ is quasicontinuous at a,
- $a = \infty$: in this case for each open neighbourhood U of a and open neighbourhood V of $g_{\mathfrak{S}}(a) = u_1$, by Remark 3.1 there exists finite subset H of $\mathcal{K}$ such that $U = A(\mathcal{K}^n) \setminus H$, therefore there exists $p, q \geq 1$ such that $(2p+1, \dots, 2p+1) = u_q \in U$ in particular $W := \{(2p+1, \dots, 2p+1)\}$ is a nonempty open subset of U and
$$g_{\mathfrak{S}}(W) = \{g_{\mathfrak{S}}((2p+1, \dots, 2p+1)\} = \{u_1\} = \{g_{\mathfrak{S}}(\infty)\} \subseteq V \ .$$
Thus $g_{\mathfrak{S}}$ is quasicontinuous at $a = \infty$ in this case.

Using the above cases $g_{\mathfrak{S}} : \mathcal{K}^n \to X$ is quasicontinuous.

Thus
$$\begin{aligned} card(Q(A(\mathcal{K}^n), X)) &\geq card\{g_{\mathfrak{S}} : \mathfrak{S} \text{ is a sequence in } X\} \\ &= card\{\mathfrak{S} : \mathfrak{S} \text{ is a sequence in } X\} \\ &= card(X^{\mathbb{N}}) = card(X)^{\aleph_0} \ . \end{aligned}$$
Using a similar method described in Theorem 2.2 completes the proof. □

4. Conclusion

For Khalimsky line $\mathcal{K}$, Khalimsky plane $\mathcal{K}^2$, Khalimsky circle $A(\mathcal{K})$, Khalimsky sphere $A(\mathcal{K}^2)$ and topological space X we show the collection of all quasicontinuous maps from $\mathcal{K}$ (resp $\mathcal{K}^2$, $A(\mathcal{K})$, $A(\mathcal{K}^2)$) to X has $card(X)^{\aleph_0}$ elements. In particular for countable X with at least two elements, $Q(\mathcal{K}, X)$ (the collection of all quasicontinuous maps from $\mathcal{K}$ to X) is uncountable.

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A NOTE ON THE ALMOST CP-SPACES

SOMAYEH SOLTANPOUR, MEHRDAD NAMDARI, AND SAHAM MAJIDIPOUR

ABSTRACT. Let $C_c(X)$ be the functionally countable subalgebra of $C(X)$. Almost CP-spaces investigate algebraically and topologically and we characterize some equivalent conditions with almost CP-spaces.

Key words and phrases: CP-space; almost CP-point; almost CP-space.

1. Introduction

Let $C(X)$ ($C^*(X)$) be the ring of real-valued continuous (bounded) functions on a space X. All topological spaces in this article are Tychonoff. The subalgebra $C^*(X)$ of $C(X)$ has an important role in the study the relation between topological properties of X and algebraic properties of $C(X)$. It is shown that, for any topological space X, $C^*(X) \cong C(\beta X)$, (βX is the Stone$-\check{C}$ech compactification of X). So $C^*(X)$ is a type of $C(X)$. Karamzadeh et al. introduced and studied the subring $C_c(X)$ of $C(X)$, consisting real-valued continuous with countable image and turns out that $C_c(X)$ is not isomorphic to any $C(Y)$ in general, see [3], [2], and [1]. For each $f \in C_c(X)$, zero-set of f, denotes by $Z_c(f)$ and $X \setminus Z_c(f) = coz_c(f)$ is the cozero-set of f. The set of all zero-sets of f (cozero-sets of f) is denoted by $Z_c(X)$ ($Coz_c(X)$). All topological space X that have a base of clopen sets is called zero dimensional. Banaschewski has shown that every zero dimensional space X has a zero dimensional compactification, denoted by $\beta_0 X$. υX denotes the Hewit real compactification and for a zero dimensional space X the counterpart of υX is $\upsilon_0 X$. A subset S of X is called C^*-embedded in X if for each $f \in C_c^*(S)$ there exists $\bar{f} \in C_c^*(X)$ such that $\bar{f}|_S = f$. A space X in Y is Z_c-embedded if for each $Z \in Z_c(X)$, there exists a set H in $Z_c(Y)$ such that $H \cap X = Z$. We recall that $M_{cp} = \{f \in C_c(X) : p \in Z_c(f)\}$ and $O_{cp} = \{f \in C_c(X) : p \in \operatorname{int}_X(Z_c(f))\}$. $p \in X$ is called an almost CP-point if for each $f \in M_{cp}$, $\operatorname{int}_X(Z_c(f)) \neq \emptyset$. X is called an almost CP-space where each point of X is an almost CP-point. Clearly each almost P-space is almost CP-space but the converse is not necessarily true. $\mathbb{R}$ with the usual topology is not an almost P-space but it is a CP-space and consequently it is an almost CP-space. It is shown that each zero dimensional almost CP-space is almost P-space.

2. Almost CP-spaces

In this section we investigate almost CP-spaces topologically and algebraically.

Proposition 2.1. *If X is a countably completely regular and almost CP-space, then each nonempty G_δ-set in X has a nonempty interior.*

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In the next proposition the equivalent conditions with almost CP-space are characterized for $C_c(X)$.

Proposition 2.2. *Let X be a countably completely regular space, then the following statements are equivalent.*

(1) *X is an almost CP-space.*
(2) *For each $f \in C_c(X)$, if $Z_c(f) \neq \emptyset$, then $\mathrm{int}_X Z_c(f) \neq \emptyset$.*
(3) *For each $g \in C_c(X)$, $Z_c(g)$ is a regular-closed set (i.e., $Z_c(g) = \mathrm{cl}_X \mathrm{int}_X Z_c(g)$).*
(4) *If A is a G_δ-set, then $\mathrm{int}_X A$ is dense in A.*
(5) *Each non invertible element in $C_c(X)$ is a zero divisor.*
(6) *For each $f \in C_c(X)$, there exists $h \neq 1$ such that $f = h^2 f$.*
(7) $\bigcup_{p\in X} O_{cp} = \bigcup_{p\in X} M_{cp}$.

Lemma 2.3. *Let X be an almost CP-space where T is a dense subset of X and Z is a zero-set in X. Then $\mathrm{cl}_X(\mathrm{int}_X(Z)) = \mathrm{cl}_X(\mathrm{int}_X(Z \cap T))$.*

Definition 2.4. *An element $f \in C_c(X)$ is called a regular element (non-zero divisor) in $C_c(X)$ if $fg = 0$ and $g \in C_c(X)$ implies that $g = 0$, equivalently $\mathrm{int}_X Z_c(f) = \emptyset$. An ideal I of $C_c(X)$ is called regular if it contains a regular element. f is regular in $C_c(X)$ if and only if $X \setminus Z_c(f) = coz_c(f)$ is dense in X.*

Theorem 2.5. *Let X be a topological space, then the following statements are equivalent.*

(1) *X is an almost CP-space.*
(2) *For a zero dimensional space X, $\upsilon_0 X$ is an almost CP-space.*
(3) *Each dense Z_c-embedded subspace of X is C_c-embedded in X.*
(4) *Each regular element in $C_c(X)$ has an inverse element in $C_c(X)$.*
(5) *$C_c(X)$ has no proper regular ideal.*

Corollary 2.6. *Let X and Y are topological spaces and $C_c(X) \cong C_c(Y)$. If X is an almost CP-space, then Y is also an almost CP-space.*

From [5], we know that a zero dimensional space X is pseudocompact if and only if $\beta_0 X = \upsilon_0 X$, so we have the next corollary.

Corollary 2.7. *Let X be a zero dimensional topological space, then $\beta_0 X$ is an almost CP-space if and only if X is a pseudocompact and almost CP-space.*

We recall that an ideal I in a ring R is a z°-ideal if it consists of zero divisors and for each $a \in I$, $P_a \subseteq I$, where P_a is the intersection of all the minimal prime ideal of R containing a. The z°-ideal in $C_c(X)$ is denoted by z_c°-ideal. Clearly each ideal P_a for any $a \in C_c(X)$ is a z_c°-ideal, which is called basic z_c°-ideal. The next result, which is an algebraic characterization of almost CP-spaces, immediately shows that the sum of z_c°-ideals in $C_c(X)$, where X is an almost CP-space, is either a z_c°-ideal or the whole of $C_c(X)$.

Theorem 2.8. *The following statements are equivalent.*

(1) *X is an almost CP-space.*
(2) *Every z_c-ideal in $C_c(X)$ is a z_c°-ideal.*
(3) *Every maximal ideal (resp., prime z_c-ideal) in $C_c(X)$ is a z_c°-ideal.*
(4) *Every maximal ideal in $C_c(X)$ consists entirely of zero divisors.*

(5) *The sum of any two ideals consisting of zero divisors is either $C_c(X)$ or consists of zero divisors.*
(6) *For each non-unit element $f \in C_c(X)$, there exists a nonzero element $g \in C_c(X)$ with $P_f \subseteq Ann(g)$.*

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The 12th Seminar on Geometry and Topology
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A NOTE ON ALMOST LOCALLY λ-COMPACT SPACES

SOMAYEH SOLTANPOUR, MEHRDAD NAMDARI, AND SAHAM MAJIDIPOUR

Abstract. Let X be a Hausdorff topological space if every nonempty open set of X contains a nonempty set with λ-compact closure, X is called almost locally λ-compact. It is shown that every locally λ-compact P_λ-space is an almost locally μ-compact space, for some $\mu \leq \lambda$. It turns out that a locally λ-compact and λ-pseudo discrete space X is a λ-discrete space.

Key words and phrases: λ-compact; locally λ-compact; λ-pseudo discrete.

1. Introduction

Throughout this article all topological spaces X are infinite completely regular Hausdorff (i.e., Tychonoff) unless otherwise mentioned and all nonzero ring homomorphisms carry the identity to the identity. We recall that a topological space X (not necessarily Hausdorff) is said to be λ-compact whenever each open cover of X can be reduced to an open cover of X whose cardinality is less than λ, where λ is the least infinite cardinal number with this property. We note that compact spaces (resp., Lindelöf noncompact spaces) are $\aleph_0$-compact (resp., $\aleph_1$-compact spaces) and in general every topological space X is λ-compact for some infinite cardinal number λ. It is also observed, in [4] that given any infinite cardinal number λ there exists a space Y which is λ-compact and if $\lambda \geq \aleph_1$ is a regular cardinal, then Y is a P-space too (note, there are no infinite compact P-spaces). We recall that an ideal $I$of $C(X)$ with $g(I) \geq \lambda$ to be λ-fixed, where λ is an infinite cardinal number, whenever each subideal A of I with $g(A) < \lambda$, is fixed, see [4]. In [6], it turns out that X is a λ-compact space if and only if every λ-fixed ideal in $C(X)$ is fixed and λ is the least infinite cardinal number with this property. Consequently, X is compact if and only if every $\aleph_0$-fixed ideal is fixed. Let λ be any infinite cardinal number and X be a topological space. We remind the reader that X is called a P_λ-space if whenever $\{G_i : i \in I\}$ with $|I| < \lambda$ is a collection of open sets in X, then $G = \bigcap_{i\in I} G_i$ is open too. Clearly every topological space is a $P_{\aleph_\circ}$-space, and X is a P-space if and only if X is a $P_{\aleph_1}$-space. Almost locally λ-compact spaces are introduced and investigated. Let X be a topological space, we put $C_{K,\lambda}(X) = \{f \in C(X) : \ cl(X \setminus Z(f)) \text{ is } \lambda\text{-compact}\}$. In particular, $C_{K,\aleph_0}(X) = C_K(X)$. It is shown that $C_{K,\lambda}(X)$ is a free ideal if and only if X is locally λ-compact but not compact. The reader is referred to [3] for undefined terms and notations.

2. Main results

Recall that a topological space X is called λ-compact if each open cover of X can be reduced to an open cover whose cardinality is less than λ, where λ is the least infinite cardinal number with this property, see [6].

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Definition 2.1. *A Hausdorff space X is called almost locally λ-compact if every nonempty open set of X contains a nonempty set with λ-compact closure.*

We remind the reader that almost locally $\aleph_0$-compact space is called almost locally compact space, see [2]. A Hausdorff space X is said to be locally λ-compact if every point in X has a λ-compact neighborhood, see [6]. If $\mu \leq \lambda$, then every μ-compact subspace of a P_λ-space is closed, see [6].

Proposition 2.2. *Every locally λ-compact P_λ-space is an almost locally μ-compact space, for some $\mu \leq \lambda$.*

Proof. Suppose that X is a locally λ-compact space, then for every $x \in X$ there exists a λ-compact neighborhood U_x. If A is a nonempty open set of X and $x \in A$, there exists an open neighborhood V_x that containing x where $x \in V_x \subseteq A$. Put $G = U_x \cap V_x$ so $x \in G$ and G is an open subset of A. Since U_x is a λ-compact set of a P_λ-space X we infer that U_x is closed. $cl(G) \subseteq cl(U_x) = U_x$ and U_x is λ-compact, so $cl(G)$ is μ-compact for some $\mu \leq \lambda$ and the proof is complete. □

Let X be almost locally λ-compact and Y be an open subspace of X. If A be a nonempty open subset of Y, then A is open in X and consequently there exists a nonempty set G where $cl_X(G)$ is λ-compact and $G \subseteq A$. $cl_Y(G) \subseteq cl_X(G)$, so $cl_Y(G)$ is μ-compact for some $\mu \leq \lambda$. Now, we can give the following fact.

Proposition 2.3. *Every open subspace of an almost locally λ-compact space is an almost locally μ-compact space for some $\mu \leq \lambda$.*

Definition 2.4. *A topological space X is said to be a λ-pseudo discrete space if every λ-compact subset of X has interior of cardinality less than λ, where λ is the least infinite cardinal number with this property.*

$\aleph_0$-pseudo discrete space is called pseudo discrete space, see [1]. Clearly, if every λ-compact subset of X has cardinality less than λ then X is a λ-pseudo discrete space and in particular, every P-space is a pseudo discrete space. The next two results are the counterparts of facts in [2].

Proposition 2.5. *Every open subspace of a λ-pseudo discrete space is λ-pseudo discrete space.*

Proof. Suppose that X is a λ-pseudo discrete space and Y is an open subspace of X. If A is a λ-compact subset of Y, then A is a λ-compact subset of X and $|int_X(A)| < \lambda$. We know $int_X(A) = int_Y(A) \cap int_X(Y) = int_Y(A)$ hence the proof is complete. □

We recall that a point $x \in X$ is called λ-isolated if it has a neighborhood with cardinality less than λ, and $I_\lambda(X)$ is denoted the set of λ-isolated points of X. A topological space X is said to be λ-discrete if every point of X is λ-isolated i.e., $I_\lambda(X) = X$, see [6].

Proposition 2.6. *Every locally λ-compact and λ-pseudo discrete space is a λ-discrete space.*

Proof. Since X is locally λ-compact we infer that $x \in X$ has a neighborhood V where $cl(V)$ is λ-compact. The definition of λ-pseudo discrete implies that the interior of $cl(V)$ has cardinality less than λ and consequently $|V| < \lambda$ i.e., x has a neighborhood with cardinality less than λ. Therefore X is λ-discrete. □

Definition 2.7. *A topological space X is said to be a λ-pseudo space if every λ-compact subset of X has cardinality less than λ.*

We recall that $\aleph_0$-pseudo space is called pseudo finite space, see [5]. Clearly, every subspace of a λ-pseudo space is a λ-pseudo space.

Example 2.8. *Every λ-pseudo space is λ-pseudo discrete but the converse is not true in general. For instance, we consider the free union of a discrete space D and the rational numbers set $\mathbb{Q}$, it is a pseudo discrete space which is not pseudo finite.*

We put $C_{K,\lambda}(X) = \{f \in C(X) : \ cl(X \setminus Z(f)) \text{ is } \lambda\text{-compact}\}$. In particular, $C_{K,\aleph_0}(X) = C_K(X)$.

Theorem 2.9. *$C_{K,\lambda}(X)$ is a free ideal if and only if X is locally λ-compact but not compact.*

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ON THE REAL MAXIMAL IDEALS OF $L_{cc}(X)$

SOMAYEH SOLTANPOUR

ABSTRACT. Let $L_{cc}(X)$ denote the locally countable subalgebra of $C(X)$ whose local domain is cocountable. We investigated the real maximal ideals of $L_{cc}(X)$. For a maximal ideal M in $L_{cc}(X)$ it is shown that M is real if and only if $Z_{lc}[M]$ is closed under countable intersection if and only if $Z_{lc}[M]$ has the countable intersection property.

Key words and phrases: Local domain; real; infinitely large; infinitely small.

1. Introduction

In [2] and [1] the subalgebra $C_c(X)$ ($C_c^*(X)$) of $C(X)$, consisting of all (resp., bounded) functions with countable image is introduced and studied. In contrast to $C^*(X)$, $C_c(X)$ enjoys some nice algebraic properties of $C(X)$, which are not usually satisfied by $C^*(X)$. Motivated by the fact that $C_c(X)$ is the largest subring of $C(X)$ whose elements have countable image, the subring $L_c(X)$ of $C(X)$ which lies between $C_c(X)$ and $C(X)$ is introduced in [4]. Let $f \in C(X)$, then its local domain, which is denoted by C_f, is defined by $C_f = \bigcup\{U|U$ is open in X and $|f(U)| \leq \aleph_0\}$. We recall that $L_c(X)$ is the ring of all continuous functions that its local domains are dense in X. This subring naturally leads us to consider a new subring of $C(X)$, namely $L_{cc}(X)$, which lies between $C_c(X)$ and $L_c(X)$. Our aim in this article is to study $L_{cc}(X)$ further and try to record some facts about $L_{cc}(X)$ and indicate the relations between topological properties of X and the algebraic properties of $L_{cc}(X)$. Let $L_{cc}(X) = \{f \in C(X) : |X \setminus C_f| \leq \aleph_0\}$, where C_f is the union of all open subsets $U \subseteq X$ such that $|f(U)| \leq \aleph_0$. In [5], it is shown that $L_{cc}(X)$ enjoys most of the important properties which are shared by $C(X)$ and $C_c(X)$. For a topological space X, we denote by $Z_{lc}(X)$ the set of all zero-sets of $L_{cc}(X)$. Whenever $C(X)/M^p \cong \mathbb{R}$, then M^p is called real, else hyper-real and υX is in fact the set of all $p \in \beta X$ such that M^p is real. For an element f of $C(X)$, the zero-set (resp., cozero-set) of f is denoted by $Z(f)$ (resp., $Coz(f)$) which is the set $\{x \in X : f(x) = 0\}$ (resp., $X \setminus Z(f)$). We use $Z(X)$ (resp., $Coz(X)$) to denote the collection of all the zero-sets (resp., cozero-sets) of elements of $C(X)$. Similarly, $Z_{lc}(X)$ (resp., $Coz_{lc}(X)$) is denoted the set $\{Z(f) : f \in L_{cc}(X)\}$ (resp., $\{Coz(f) : f \in L_{cc}(X)\}$). A zero-dimensional topological space is a Hausdorff space with a base consisting of clopen sets. We refer to [3], [2], and [1] for any notation and terminology unfamiliar to the reader. Throughout this article all topological spaces are assumed to be infinite completely regular and Hausdorff (unless otherwise mentioned).

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2. Main results

Definition 2.1. [5, Definition 3.1] *Let $f \in C(X)$ and C_f be the union of all open sets $U \subseteq X$ such that $f(U)$ is countable, i.e.,*

$$C_f = \bigcup \{U | U \text{ is open in } X \text{ and } |f(U)| \leq \aleph_0\}.$$

We call C_f the local domain of f and denote by $L_{cc}(X)$ the set of all $f \in C(X)$ whose local domain is cocountable, i.e.,

$$L_{cc}(X) = \{f \in C(X) : |X \backslash C_f| \leq \aleph_0\}.$$

It is obvious that $L_{cc}(X)$ is a subring of $C(X)$ containing $C_c(X)$. In fact $L_{cc}(X)$ is a subalgebra as well as a sublattice of $C(X)$ and we call it the *co-locally functionally countable subalgebra of $C(X)$*.

We remind the reader that a Hausdorff space X is called co-locally countable completely regular (briefly, *lcc*-completely regular) if whenever $F \subseteq X$ is a closed set and $x \in X \setminus F$, then there exists $f \in L_{cc}(X)$ with $f(F) = 0$ and $f(x) = 1$, see [5].

It is shown that in studying $L_{cc}(X)$ the space can be consider co-locally countable completely regular.

Definition 2.2. [5, Definition 3.6] *Let $f \in C(X)$ and C_f^F be the union of all open sets $U \subseteq X$ such that $f(U)$ is finite, i.e.,*

$$C_f^F = \bigcup \{U | U \text{ is open in } X \text{ and } |f(U)| < \aleph_0\}.$$

Denote by $L_{cF}(X)$ the set of all $f \in C(X)$ such that C_f^F is cocountable, and call it co-locally functionally finite subalgebra of $C(X)$, i.e.,

$$L_{cF}(X) = \{f \in C(X) : |X \backslash C_f^F| \leq \aleph_0\}.$$

In a special case, for $f \in C(X)$ let C_f^c be the union of all open sets $U \subseteq X$ such that $f(U)$ is constant, i.e.,

$$C_f^c = \bigcup \{U|\ U \text{ is open in } X \text{ and } |f(U)| = 1\}.$$

We define $L_{c1}(X)$ to be the set of all $f \in C(X)$ such that C_f^c be cocountable in X, and call it co-locally functionally constant subalgebra of $C(X)$, i.e.,

$$L_{c1}(X) = \{f \in C(X) : |X \backslash C_f^c| \leq \aleph_0\}.$$

Clearly, $L_{cF}(X)$ and $L_{c1}(X)$ are subalgebra of $L_{cc}(X)$.

Definition 2.3. *A maximal ideal M in $L_{cc}(X)$ is called real if $\frac{L_{cc}(X)}{M} \cong \mathbb{R}$ and if not real, it is called hyper-real.*

Proposition 2.4. *Let M be a maximal ideal in $L_{cc}(X)$ and $f \in L_{cc}(X)$. Then $|M(f)|$ is infinitely large if and only if f is unbounded on every zero-set of $Z_{lc}[M]$.*

Proposition 2.5. *Let $f \in L_{cc}(X)$ then f is unbounded on X if and only if $|M(f)|$ is infinitely large for some maximal ideal M in $L_{cc}(X)$.*

Proposition 2.6. *Let X be zero-dimensional, then f is unbounded on every non-compact zero-set of $Z_{lc}(X)$ if and only if $|M(f)|$is infinitely large for every free maximal ideal M in $L_{cc}(X)$.*

Proposition 2.7. *If M is a maximal ideal in $L_{cc}(X)$, then M is hyper-real if and only if $|M(f)|$ is infinitely small for some $f \in L_{cc}(X)$.*

Proposition 2.8. *Every maximal ideal in $L_{cc}(X)$ and $L_{cF}(X)$ is real.*

Proposition 2.9. *Every fixed maximal ideal in $L_{cc}(X)$ is real.*

We remind the reader that $L^*_{cc}(X) = C^*(X) \cap L_{cc}(X)$.

Proposition 2.10. *If M is real maximal ideal in $L_{cc}(X)$, then $M \cap L^*_{cc}(X)$ is a real maximal ideal in $L^*_{cc}(X)$.*

Proposition 2.11. *If M is a maximal ideal in $L^*_{cc}(X)$, then $M \cap L_{cF}(X)$ is a real maximal ideal in $L_{cF}(X)$.*

An ideal M in $L^*_{cc}(X)$ is maximal if and only if it is a contraction of a maximal ideal in $C^*(X)$. Consequently the maximal ideals in $L^*_{cc}(X)$ are $M^{*p}_{lc} = \{f \in L^*_{cc}(X) : f^\beta(p) = 0\}$ where $p \in \beta X$.

Proposition 2.12. *Let M be a maximal ideal in $L_{cc}(X)$, then M is real if and only if $M \cap L^*_{cc}(X)$ is a maximal ideal in $L^*_{cc}(X)$.*

Proposition 2.13. *Let M be a maximal ideal in $L_{cc}(X)$, then M is real if and only if $Z_{lc}[M]$ is closed under countable intersection if and only if $Z_{lc}[M]$ has the countable intersection property.*

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GENERALIZED POWER SERIES FIELD WITH ORDER TOPOLOGY

PARISA ABBASPOUR AND JAFAR SADEGH EIVAZLOO

Abstract. We will be dealing with the ordered field of generalized power series, $[[\mathbb{R}^{\mathbb{Z}}]]$, and will prove that the field of real rational functions, $\mathbb{R}(x)$, is dense in it. As a corollary, the field $\mathbb{R}(x)$ is not dense in its real closure.

Key words and phrases: generalized power series; real closed field, Scott complete, real closure.

1. Introduction

For any ordered field F and ordered abelian group G, the set $[[F^G]]$ of all functions $G \to F$ whose supports are well ordered in G equipped with pointwise sum and Cauchy product $(f_1f_2)(g) = \sum_{i+j=g} f_1(i)f_2(j)$ (a finite sum by the condition on the supports) forms a field. It can be ordered by comparison of values at the minimum of support of the difference. Elements of $[[F^G]]$ can also be thought of as those formal power series $\sum_{g\in G} f(g)t^g$ which have well ordered supports. The indeterminate t is taken to be a positive F-infinitesimal. Indeed,

$$[[F^G]] = \left\{ \sum_{g\in G} a_g t^g \mid \text{the support } \{g|a_g \neq 0\} \text{ of} \sum_{g\in G} a_g t^g \text{ is well ordered} \right\}.$$

It is ordered by $\sum_{g\in G} a_g t^g > 0$ if $a_{g_0} > 0$, where g_0 is the minimum of support of $\sum_{g\in G} a_g t^g$. Then, $[[F^G]]$ is equipped with the interval topology induced by the defined order.

An ordered field R is called real closed if every nonnegative element of R has a square root in R, and for every $P(x) \in R[x]$ with odd degree, $P(x) = 0$ has a solution in R. For every ordered field F, there is an ordered field E which is the unique real closed ordered field that extents F. This real closed field which is called the real closure of F, is denoted by $RC(F)$. The real closure of the ordered rational field $\mathbb{Q}$ is the real algebraic numbers $\widetilde{\mathbb{Q}}$. By [2] (6.10), the generalized power series field $[[F^G]]$ is real closed if and only if F is so and G is divisible. For example, $[[\mathbb{R}^{\mathbb{Q}}]]$ is real closed, while the Laurent series ordered field $[[\mathbb{R}^{\mathbb{Z}}]]$ is not.

A cut of an ordered field F is a subset $C \subset F$ which is downward closed in it. A cut C is a gap if it does not have a least upper bound in the field. An ordered field without any gap is called Dedekind complete. A gap $C \subset F$ is called regular if for any $\epsilon \in F^{>0}$, there exist $a \in C$ and $b \in F \setminus C$ such that $b - c < \epsilon$. An ordered field without any regular gap is called Scott complete. It was proved in [3] (Theorem 1), that any ordered field F has a (unique up to an isomorphism of ordered fields which

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is identity on F) Scott completion. It is characterized by being Scott complete and having F dense in it.

We recall the following theorem from [1].

Theorem 1.1. *For any ordered abelian group G and ordered field F, the generalized power series field $[[F^G]]$ is Scott complete if and only if F is so.*

For example, the Laurent series ordered field $[[\mathbb{R}^{\mathbb{Z}}]]$ is Scott complete.

Furthermore, the real closure and Scott completion of an ordered field F are related by the following theorem.

Theorem 1.2. *The ordered field F is dense in $RC(F)$ if and only if its Scott completion is real closed.*

For proof, see [3] (Theorem 2).

2. Main results

Definition 2.1. *Let F be a field and G an ordered abelian group. A valuation on a field F is a function $v : F \to G \cup \{\infty\}$ with following properties.*

(1) $v(a) = 0$ *if and only if* $a = \infty$,
(2) $v(ab) = v(a) + v(b)$,
(3) $v(a + b) \geq \min\{v(a), v(b)\}$, *with equality if* $v(a) \neq v(b)$.

Then, F is called a valued field with the valuation v and the valuation group G.

Remark 2.2. *Recall that the generalized power series field $[[F^G]]$ is a valued field with the valuation $v(\sum_{g\in G} a_g t^g) = g_0$, where g_0 is the minimum of all $g \in G$ such that $a_g \neq 0$.*

The field of real rational functions, $\mathbb{R}(t)$, is an ordered field with the order $<$ induced from $(\mathbb{R}, <)$ and $t < \mathbb{R}$ as a positive infinitesimal. This ordered field is an ordered subfield of the Laurent series ordered field $[[\mathbb{R}^{\mathbb{Z}}]]$. In the following theorem, we show that $\mathbb{R}(t)$ is dense in $[[\mathbb{R}^{\mathbb{Z}}]]$.

Theorem 2.3. $\mathbb{R}(t)$ *is dense in* $[[\mathbb{R}^{\mathbb{Z}}]]$.

Proof. Let $a = \sum_{g\in G} a_g t^g$ and $b = \sum_{g\in G} b_g t^g$ be two elements in $[[\mathbb{R}^{\mathbb{Z}}]]$ such that $0 < a < b$. Then we have the following two cases.

Case i) $v(b) < v(a)$. In this case, $b_{v(b)} > 0$ and for every element $r \in \mathbb{R}$ such that $b_{v(b)} > r > 0$ we have $a < rt^{v(b)} < b$. Notice that $rt^{v(b)} \in \mathbb{R}(t)$.

Case ii) $v(b) = v(a) := g$. In this case, $0 < a_{v(a)} < b_{v(b)}$. Then, for every element $r \in \mathbb{R}$ with $a_{v(a)} < r < b_{v(b)}$, we have $a < rt^g < b$ while rt^g is in $\mathbb{R}(t)$. $\square$

It is clear that $\mathbb{R}(t)$ is a proper subfield of $[[\mathbb{R}^{\mathbb{Z}}]]$. So there exists an element $a \in [[\mathbb{R}^{\mathbb{Z}}]] \setminus \mathbb{R}(t)$. Then the set $C := \{b \in \mathbb{R}(t) | b < a\}$ is a gap in $\mathbb{R}(t)$. As $\mathbb{R}(t)$ is dense in $[[\mathbb{R}^{\mathbb{Z}}]]$, the gap C is regular. So the ordered field $\mathbb{R}(t)$ is not Scott complete. Indead, we have the following conclusion.

Corollary 2.4. *The Scott completion of the ordered field $\mathbb{R}(t)$ is the Laurent series ordered field $[[\mathbb{R}^{\mathbb{Z}}]]$.*

Proof. As $\mathbb{R}(t)$ is dense in $[[\mathbb{R}^{\mathbb{Z}}]]$ which is Scott complete by Theorem 1.1, $[[\mathbb{R}^{\mathbb{Z}}]]$ is the Scott completion of $\mathbb{R}(t)$. $\square$

Corollary 2.5. *The ordered field $\mathbb{R}(t)$ is not dense in its real closure.*

Proof. The Laurent series ordered field $[[\mathbb{R}^{\mathbb{Z}}]]$ is not real closed, e.g. the positive infinitesimal t does not have its square root in $[[\mathbb{R}^{\mathbb{Z}}]]$. So the Scott completion of $\mathbb{R}(t)$ is not real closed. Thus, by Theorem 1.2, $\mathbb{R}(t)$ is not dense in its real closure. □

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FINDING GENERALIZED SYMMETRIES OF NC-BURGERS EQUATIONS USING MASTER'S SYMMETRY.

MEHDI JAFARI AND MOJDEH GANDOM

Abstract. In this paper, we propose using generalized symmetries as an alternative to recursion operators for infinite hierarchies of symmetries in evolution equations. "Master symmetries" are generalized vector fields that produce new symmetries when combined with existing ones. They can be used to solve problems like the non-commutative Burgers equations.

Key words and phrases: Master symmetry ;nc-Burgers ; Generalized Symmetries.

1. Introduction

Basically, the calculation of generalized symmetries of a certain system of differential equations is done like the calculation of geometric symmetries, but with this process that we must first put the symmetry in an evolutionary form. Then the order of derivatives on which the characteristic $Q(x, u^{(n)})$ may depend must be predetermined. The basic trade-off in this context is that the more derivatives of u on which Q depends, the more generalized symmetries are found, but on the other hand, this solution is long and tedious, whereas for surface symmetries higher will be more boring and it takes time. Because the set of generalized symmetries is a non-degenerate system of differential equations, it forms a Lie algebra. Use this result to construct new symmetries and obtain the master symmetry for non-commutative Burgers equations.

2. Preliminaries

Let $x = (x^1, \ldots, x^p)$ independent variables, and $u = (u^1, \ldots, u^q)$ the dependent variables in our problem.

Definition 2.1. *A vector field v (as shown below) is defined as a generalized vector field in which, in addition to independent and dependent variables, their derivatives also appear.*

$$v = \sum_{i=1}^{p} \xi^i[u] \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \phi_\alpha[u] \frac{\partial}{\partial u^\alpha}, \tag{2.1}$$

such that ξ^i and ϕ_α are smooth differential functions.

Definition 2.2. *suppose v is generalized vector field, then if for every smooth solution $u = f(x)$ and*

$$\Delta_\nu[u] = \Delta_\nu(x, u^{(n)}) = 0, \quad \nu = 1, 2, \ldots, l \tag{2.2}$$

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be satisfied in the following relationship.

$$prv[\Delta_v] = 0, \quad \nu = 1, 2, \ldots, l. \tag{2.3}$$

In this case v is a generalized infinitesimal symmetry of a system of differential equations.

Definition 2.3. *assume q-tuple of differential functions such that*

$$Q[u] = (Q_1[u], \ldots, Q_q[u])\epsilon\mathcal{A}^q.$$

then the generalized vector field v_Q is called an evolutionary vector field if and only if v_Q be defined as follows, and Q is called its characteristic

$$v_Q = \sum_{\alpha=1}^{q} Q_\alpha[u]\frac{\partial}{\partial u^\alpha}. \tag{2.4}$$

Theorem 2.4. *If v is a symmetry of a system of differential equations, then v_Q is its evolutionary representative.*

3. third order Generalized Symmetries

Consider non-commutative Burgers equation as follows:

$$u_t = u_{xx} + 2uu_x. \tag{3.1}$$

We want to calculate all generalized third-order symmetries of the above equation Considering the infinitesimal generator in evolutionary form $v = Q\partial_u$ and considering $Q = Q(x, t, u, u_x, u_{xx}, u_{xxx})$ and according (2.2) to we will have

$$D_tQ = D_x^2Q + 2uD_xQ + 2u_xQ. \tag{3.2}$$

By using (3.1) and its prolongations, we can substitute any t derivatives of u in solutions .i.e

$u_t = u_{xx} + 2uu_x,$
$u_{tx} = 2u_x^2 + 2uu_{xx} + u_{xxx},$
$u_{tt} = 8u_{xx}u_x + 8uu_x^2 + 4u^2u_{xx} + 4uu_{xxx} + u_{xxxx},$
$u_{txx} = 6u_{xx}u_x + 2uu_{xxx} + u_{xxxx},$
$\vdots$

After examining (3.2), we can identify u's derivative coefficients in descending order. Finally we conclude that every third order generalized symmetry of the nc-Burgers' equation has a linear, constant-coefficient combination of seven basic characteristics:

$Q_1 = (uu_{xx} + 1/3u_{xxx} + u^2u_x + u_x^2)t^3 + \cdots,$

$Q_2 = -3/4 + (3u_x^2 + 3u^2u_x + 3uu_{xx} + u_{xxx})t^2 + (2xuu_x + u^2 + xu_{xx})t + 1/4u_xx^2 + 1/2xu,$

$Q_3 = 1/2 + (6u_x^2 + (6u^2 + 1)u_x + 6uu_{xx} + 2u_{xxx})t + 2xuu_x + u^2 + xu_{xx},$

$Q_4 = (2uu_x + u_{xx})t^2 + (1/2xu_x + 1/2u)t + u_x + 1/4x,$

$Q_5 = (2uu_x + u_{xx})t + 1/2xu_x + 1/2x,$

$Q_6 = u_{xx} + 2uu_x,$

$Q_7 = uu_{xx} + 1/3u_{xxx} + u^2u_x + u_x^2.$

4. The Lie Bracket

Definition 4.1. *Let v_Q and v_R be evolutionary vector fields. Their Lie bracket $[v_Q, v_R] = v_S$ is an evolutionary vector field with characteristic.*

$$S = prv_Q(R) - prv_R(Q). \tag{4.1}$$

Theorem 4.2. *Generalized symmetries of a system of differential equations form a Lie algebra.*

5. Forth-order generalized symmetries

In certain cases, this result (i.e (4.2)) can be used to construct new generalized symmetries (Forth-order generalized symmetries) from known ones in (third-order generalized symmetries) Therefore, given that $[v_i, v_j]$ is a symmetry with characteristic $prv_i(Q_j) - prv_j(Q_i)$ for any i, j. So for nc-Burgers will have

$$\begin{aligned} Q_8 &= prv_3(Q_7) - prv_7(Q_3), \quad Q_9 = prv_2(Q_7) - prv_7(Q_2),\\ Q_{10} &= prv_1(Q_7) - prv_7(Q_1), \quad Q_{11} = prv_1(Q_3) - prv_3(Q_1),\\ Q_{12} &= prv_1(Q_2) - prv_2(Q_1) = prv_2(Q_3) - prv_3(Q_2). \end{aligned}$$

In such a way that the characteristics of $Q_8, Q_9, Q_{10}, Q_{11}, Q_{12}$ make the new fourth order symmetries for nc-Burger as follows

$$\begin{aligned} Q_8 =& 1/2u_{xx} + uu_x + u_{xxxx} + 10u_{xx}u_x + 4uu_{xxx} + 12uu^2 + 6u^2u_x + 4u^3u_x,\\ Q_9 =& 4tu^3u_x + tu_{xxxx} + 6u_{xx}tu^2 + 12tuu_x^2 + 10tuu_xu_{xx} + \cdots,\\ Q_{10} =& 1/4u + 1/8u_{xx}x^2 + 1/2t^2u_{xxxx} + 1/2tu^3 + 2t^2u^3u_x + \cdots,\\ Q_{11} =& 1/4x + 2t^3u_{xxxx} + 3t^2u^3 + 1/4x^3u_x + 3/4x^2u + 8xt^3u^3 + \cdots,\\ Q_{12} =& 5/2t^3u_{xx} + 3t^3uu_{xx} + 3/2t^2x^2uu_x + 3t^3u^2u_xx + 3/2ut^2 + 3u_xxt^2 + \cdots. \end{aligned}$$

This process can be repeated indefinitely.

6. Fifth-order generalized symmetries

In a similar way, the following results are obtained for fifth-order symmetries.

$$\begin{aligned} Q_{13} &= prv_1(Q_{11}) - prv_{11}(Q_1), \qquad Q_{14} = prv_1(Q_{10}) - prv_{10}(Q_1),\\ Q_{15} &= prv_{11}(Q_3) - prv_3(Q_{11}), \qquad Q_{16} = prv_{11}(Q_7) - prv_7(Q_{11}),\\ Q_{17} &= prv_{10}(Q_7) - prv_7(Q_{10}), \qquad Q_{18} = prv_8(Q_3) - prv_3(Q_8), \end{aligned}$$

such that:

$$\begin{aligned} Q_{13} &= 5t^5u^4u_x + 10t^5u^3u_{xx} + 5t^3u^3x + 5t^5 + u_{xxxx}u + 5/2t^4u_{xxxxx} + \cdots,\\ Q_{14} &= 5t^4u_{xx}^2 + 9u_x^2t^3 + 9/4u^2t^2 + 1/2t^4u_{xxxxx} + 27/8u_xt^2 + \cdots,\\ Q_{15} &= 10t^3u_{xxxxx} + 60u^3u_xt^2x + 45/2u^2u_xtx^2 + \cdots,\\ Q_{16} &= 5/2u_x + 3t^2u_{xxxxx} + 90t^2u^2u_x^2 + 30u_{xx}t^2u^3 + \cdots,\\ Q_{17} &= 25u_xtuu_{xx} + 1/2tu_{xxxxx} + 1/4u_{xxxx}x + 5tu_{xx}^2 + 15/2tu_x^3 + \cdots,\\ Q_{18} &= 1/2u_x + 3u_{xxx} + 4u_{xxxxx} + 40u_{xx}^2 + 9u_x^2 + \cdots. \end{aligned}$$

his process can be repeated indefinitely so $Q_{19} = prv_7(Q_{13}) - prv_{13}(Q_7), \ldots$ will be a sixth order symmetry and etc. Thus nc-Burgers equation has an infinite collection of generalized symmetries depending on progressively higher and higher order derivatives of u. In Fuchssteiner's terminology, v_7, v_3 and v_1 are known as a "master symmetry" for the nc- Burger's equation.
This method finds generalized symmetries of differential equations of any order. But has the drawback that the order of derivatives must be specified beforehand for symmetry coefficients.

7. Main results

Generalized time-dependent symmetry can be an alternative to the recursive operator method for creating symmetry evolution equations.They're called "master symmetries." A master symmetry is a vector field w that, when v_Q is a generalized symmetry of the evolution equation, the Lie bracket $[w, v_Q]$ is also a symmetry. Note that any symmetry of the system satisfies this property, then to be really interesting, the master symmetry should produce new symmetries, mapping, say, the n-th member of the hierarchy of symmetries to the $(n+1)$-st one, as the nc-Burgers' case does.

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AN ALGORITHM FOR CONSTRUCTING A-ANNIHILATED ADMISSIBLE MONOMIALS IN THE DYER-LASHOF ALGEBRA

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Abstract. We present an algorithm for computing A-annihilated elements of the form $Q^I[1]$ in H_*QS^0 where I runs through admissible sequences of positive excess. This is algorithm with polynomial time complexity to address a sub-problem of an unsolved problem in algebraic topology known as the *hit problem of Peterson* which is likely to be NP-hard.

Key words and phrases: Dyer-Lashof algebra; Steenrod algebra; Λ algebra.

1. Introduction

Given a topological space X and an integer $d \geq 0$, $H^*(X;\mathbb{F}_2) = \bigoplus_{d\geq 0} H^d(X;\mathbb{F}_2)$ is a graded $\mathbb{F}_2$-algebra. For $k \geq 0$ and $d > 0$, there are $\mathbb{F}_2$-linear homomorphisms $Sq^k : H^d(X;\mathbb{F}_2) \to H^{d+k}(X;\mathbb{F}_2)$ known as Steenrod squares. These 'cohomology operations' have nice properties. In particular,

- $Sq^k(x) = 0$ if $k > d$ and $Sq^k x = x^2$ if $k = d$.
- The operation Sq^0 is just the identity.
- For $f, g \in H^*(X;\mathbb{F}_2)$, $Sq^k(fg) = \sum_{i=0}^{k} Sq^i(f)Sq^{k-i}(g)$ (Cartan formula)

There operations live in an associative and non-commutative algebra, called the (mod 2) Steenrod algebra, denoted $\mathcal{A}$. *The hit problem* is to determine $H^*(X;\mathbb{F}_2)$ is a module over $\mathcal{A}$. For the cohit module defined by

$$Q^d(H^n(X;\mathbb{F}_2)) := H^n(X;\mathbb{F}_2) \otimes_{\mathcal{A}} \mathbb{F}_2$$

the hit problem asks for determining a $\mathbb{F}_2$-basis for $Q^d(H^n(X;\mathbb{F}_2))$.

For $X(n) = \mathbb{R}P^{\times n}$ it is known that

$$P(n) := H^*(X(n);\mathbb{F}_2) \simeq \mathbb{F}_2[x_1, x_2, \ldots, x_n : \deg(x_i) = 1],$$

as an algebra. The *hit problem of Peterson* is concerned with determining generators of $P(n)$ or equivalently determining the **cohit module** $Q^d(n) := Q^d(P(n))$. This problem is open for $n > 5$ (see [5, 6, 9]). For $X = BO(n)$ it is known that

$$H^*(BO(n)) \simeq P(n)^{\Sigma_n} \simeq \mathbb{F}_2[e_i : \deg(e_i) = i, i > 0],$$

the hit problem is known as the *symmetric hit problem* which is open for $n > 4$ (see [3, 2]).

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2. Hit problem in homological setting

The hit problem is often address by determining relevant numerical invariants such as $\dim_{\mathbb{F}_2} Q^d(H^*(X;\mathbb{F}_2)$ or at least providing an upper bound in the dimension of cohit module. To study the problem in homological setting, notice that operations Sq^i induce dual operations $Sq^i_* : H_n(X;\mathbb{F}_2) \to H_{n-i}(X;\mathbb{F}_2)$ by the Universal coefficient theorem. For

$$\mathrm{Ann}_{\mathcal{A}}(H_n(X;\mathbb{F}_2)) := \left\{ x \in H_n(X;\mathbb{F}_2) | Sq^i_* x = 0 \text{ for all } i > 0 \right\}$$

we have a duality of vector spaces over $\mathbb{F}_2$ as

$$\mathrm{Hom}_{\mathbb{F}_2}(Q^d(H^n(X;\mathbb{F}_2))), \mathbb{F}_2) \simeq \mathrm{Ann}_{\mathcal{A}}(H_n(X;\mathbb{F}_2)).$$

The hit problem in dual setting is to determine the submodule of $\mathcal{A}$-annihilated classes in $H_*(X;\mathbb{F}_2)$ given by $\bigoplus_{n=1}^{+\infty} \mathrm{Ann}_{\mathcal{A}}(H_n(X;\mathbb{F}_2))$.

3. Main results

A solution to the symmetric hit problem for all n is equivalence to solving it for $X = \mathbb{Z} \times BO$ and vice versa. We have considered this point of view in [10]. We prefer study the dual of the symmetric hit problem. For $QS^0 = \mathrm{colim}\ \Omega^i S^i$, the unit of the KO spectrum provides a map $QS^0 \to \mathbb{Z} \times BO$ which induces a monomorphism of A^{op}-modules in homology. We may ask for the description of A-annihilated classes in H_*QS^0 whose complete description is unknown. But, there are some sufficient conditions that allow one to identify some of these classes. The following is due to Curtis [1, Lemma 6.2, Theorem 6.3] (see also Wellington [7, Theorem 5.6] as well as [8]).

Theorem 3.1. *For a generator $Q^I[1]$ of H_*QS^0, suppose $I = (i_1, \ldots, i_s)$ with $s > 1$ is a sequence so that $\mathrm{ex}(I) < 2^{\phi(i_1)}$ and $0 \leq 2i_{j+1} - i_j < 2^{\phi(i_{j+1})}$ for $1 \leq j \leq s-1$. Then $Q^I[1]$ is A-annihilated. If $I = (i)$ with $i < 2^{\phi(i)}$, i.e. $i = 2^t - 1$ for some $t > 0$, then $Q^i[1]$ is A-annihilated. Here, $\mathrm{ex}(Q^I x) = i_1 - (i_2 + \cdots + i_s)$.*

Here, Q^i is the i-th Kudo-Araki operations which acts on $\mathbb{F}_2$-homology of QS^0. Note that the homology of QS^0 is a polynomial algebra 'generated' by Dyer and Lashof by symbols $Q^I[1]$ where Q^I is an iterated Kudo-Araki operation given by $Q^I := Q^{i_1} \cdots Q^{i_s}$ for $I = (i_1, \ldots, i_s)$. The aforementioned result of Curtis, reduces the problem to determining all sequences I that satisfy the given conditions. We say $I = (i_1, \ldots, i_r)$ is an (indecomposable) $\mathcal{A}$-annihilated if it satisfies conditions of Theorem 3.1. Our main result is an algorithm that determines all such sequences.

Theorem 3.2. *Suppose $r > 2$ and $i_0 > 0$ are given. Consider the following algorithm.*
For $k = 0, \ldots, r-1$ do the following

(1) $n := i_k$;
(2) *choose an allowable 0 in the binary expansion of n, say n_i, and set $\phi(m) = i - 1$;*
(3) *for $j \leq \phi(m)$ set $m_j := n_{j+1}$;*
(4) *for $0 \leq j < \phi(m)$ set $m_j := 1$*
(5) $i_{k+1} := \sum_{j=0}^{\psi(i_k)-2} m_j 2^j$

Then $I = (i_1, \ldots, i_r)$ is an A-annihilated sequence. Moreover, by choosing various different allowed 0s, the above algorithm determines all such sequences. In particular, the set of A-annihilated sequence I of length r and dimension $|I| = i_0$ would be included in the set of A-annihilated sequences produced by the above algorithm.

There is a notion of an allowable 0 which we shall introduce in the next section. Here, specifically for positive integers m and n we fix that $m_j, n_j\{0,1\}$ are the coefficients of binary expansion of m and n, respectively. More precisely, $m = \sum_0^{+\infty} m_j 2^j$ and likewise n. The function ϕ is defined by $\phi(m) = \min\{j : m_j = 0\}$. It is fairly simple to compute the complexity of the above algorithm.

Corollary 3.3. *The complexity of our algorithm is $O(t^3)$. In particular, our algorithm is run in polynomial time.*

For the hit problem, the following seems of interest. Although, it is in contrast with the conjecture that the hit problem of Peterson in NP-hard.

Corollary 3.4. *(i) For every $k > 0$, there is a submodule inside*

$$\bigoplus_{n=1}^{k} \mathrm{Ann}_{\mathcal{A}}(H_n(QS^0; \mathbb{F}_2)),$$

which is determined in polynomial time.
(ii) For every $k > 0$, there is a submodule inside

$$\bigoplus_{n=1}^{k} \mathrm{Ann}_{\mathcal{A}}(H_n(\mathbb{Z} \times BO; \mathbb{F}_2)),$$

which is determined in polynomial time.

Proof. Note that our algorithm computes a submodule inside

$$\bigoplus_{n=1} \mathrm{Ann}_{\mathcal{A}}(H_n(QS^0; \mathbb{F}_2)).$$

Recall that the evident maps $QS^0 \to \mathbb{Z} \times BO$ induces a monomorphisms of $\mathcal{A}$-modules $H_n(QS^0; \mathbb{F}_2) \to H_*(\mathbb{Z} \times BO; \mathbb{F}_2)$ [10]. Applying Corollary 3.4 our claims follow. □

Finally, notice that we could define a formal evaluation from the Dyer-Lashof algebra $\mathcal{R}$ to H_*QS^0 sending Q^I to $Q^I[1]$ which is an homomorphism of A-modules. Consequently, our algorithm provides A-annihilated monomials in $\mathcal{R}$. Furthermore, noting that $\mathcal{R}$ is a quotient of the Λ algebra [8], we have a similar conclusion for monomials λ_I in the Λ algebra. We refer the reader to elsewhere for details of our conclusions.

4. Sketch of Proof for Theorem 3.2

We begin with a simple reduction.

Lemma 4.1. *For $I = (i_1, \ldots, i_r)$ let $i_0 := i_1 + \cdots + i_r$. Then I is $\mathcal{A}$-annihilated if and only if for $(i_0, I) := (i_0, i_1, \ldots, i_r)$ we have $0 < 2i_{j+1} - i_j < 2^{\phi(i_{j+1})}$ for all $j \in \{0, \ldots, r-1\}$, where $i_0 = |I|$.*

This immediately follows from the definition of the excess. Our next observations, mostly are so easy to prove once we work with binary expansions. First, we make another simple, yet useful, definition. Define $\psi : \mathbb{N} \to \mathbb{N} \cup \{0\}$ by

$$\psi(n) = \max\{j : n_j = 1\} + 1 = \min\{j : \forall k \geq j,\ n_k = 0\}.$$

The following lemma records some nice properties of ϕ and ψ.

Lemma 4.2. *Suppose $I = (i_1, \ldots, i_r)$ is an admissible sequence with $\mathrm{ex}(I) > 0$ such that $0 < 2i_{j+1} - i_j < 2^{\phi(i_{j+1})}$. Then, fixing $i_0 = \sum_{j=1}^{r} i_j$, we have*

- *I is strictly decreasing with all of its entries being odd.*
- *$\phi(i_1) \leq \cdots \leq \phi(i_r)$.*
- *For all $j \in \{2, \ldots, r\}$ we have $\psi(i_j) = \psi(i_{j-1}) - 1$.*
- *If i_0 is non-spike, then we have $\psi(i_1) = \psi(i_0) - 1$.*

Here, $k \in \mathbb{N}$ is called spike if $k = 2^t - 1$ for some $t > 0$.

Our next observation completely resolved the case when i_0 is spike.

Lemma 4.3. *(i) Suppose I is an A-annihilated sequence such that $i_0 = 2^t - 1$ for some $t > 0$. Then, $I = (2^t - 1)$.*
(ii) Suppose $I = (i_1, \ldots, i_r)$ is an A-annihilated sequence so that i_j is a spike for some j. Then, $j = r$.

So far, our results tell us that if we are given i_j then the binary expansion of i_{j+1} is somehow determined by that of i_j. The bottom line is that i_{j+1} inherits some part of the binary expansion of i_j but with a shift to the right, up to allowable 0s that are possible to choose by the algorithm. Hence, it suffices to clarify what 0s are allowable. Our next result, tells us which 0s should not be chosen, informally introducing forbidden 0s, opposite to which we have allowable 0s in our algorithm.

Lemma 4.4. *Suppose $n = \sum_{i=1}^{\psi(n)} n_i 2^i$ is a positive integer where $n_i \in \{0, 1\}$.*
(i) If $n_0 = 0$ or $n_1 = 0$ then in either case, we have a forbidden 0.
(ii) For any positive integer n, $n_{\phi(n)} = 0$ is a forbidden 0.
(iii) If n is even then $\phi(n/2) + 1$ corresponds to a forbidden 0.
(iv) Let n be even and t be the least positive integer such that for all $\phi(n/2) + 1 < j < t - 1$ we have $n_j = 0$ and $n_t = 1$. Then, for any such j, $n_j = 0$ is a forbidden 0.
(v) If m is not a spike then $\psi(n)$ corresponds to a forbidden 0.

Finally, we have our main constructive result by which we mean it allows to find the building blocks of our algorithm. We have the following.

Theorem 4.5. *Assume m and n are positive integers with binary expansions $m = \sum_j m_j 2^j$ and $n = \sum_j n_j 2^j$. If (i) For all $i \geq \phi(m)$ we have $n_{i+1} = m_i$; (ii) $\phi(n) \leq \phi(m)$ such that $\phi(n) = \phi(m)$ if and only if $n_{\phi(m)+1} = 0$ and $\phi(n) > 0$ and $\phi(n) < \phi(m)$ if and only if there exists $0 < j < \phi(m)$ such that $n_j = 0$ and $n_{\phi(m)} = 1$ and $n_{\phi(m)+1} = 0$.*
The converse also does hold, that is if the above conditions are satisfied then

$$0 < 2m - n < 2^{\phi(m)}.$$

Our algorithm now easily follows by applying this theorem iteratively.

Example 4.6. *Let $i_0 = 33$ and $r = 3$. For the binary expansion of 33 given by*

$$33 : \overbrace{\ 0\ }^{B}\ 1\ \overbrace{000}^{B}\ 01$$

we have the above 'blocks' of allowable 0s. Here, the most left 0 corresponds to $\psi(33)$ is an allowable 0. According to choices of allowable 0s we will have just two cases.

$$\begin{array}{lccccccc} i_0: & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ i_1: & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ i_2: & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ i_3: & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{array} \qquad \begin{array}{lccccccc} i_0: & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ i_1: & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ i_2: & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ i_3: & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{array}$$

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WHEN TOPOLOGICAL FUNDAMENTAL GROUPOID IS HAUSSDORFF WITH THE LASSO TOPOLOGY

ALI PAKDAMAN AND SABA DEHROOYE

Abstract. In this talk we prove that a topological fundamental groupoid (equipped with the Lasso topology) of a given space is Hausdorff when its Spanier group is trivial.

Key words and phrases: Topological Fundamental Groupoid; Lasso Topology; Spanier Group

1. Introduction

1.1. **Motivation.** Virk and Zastrow [9] have reviewed the existing topologies on the fundamental group and have studied their generalizations to the universal path space. Since fundamental groupoids are generalizations of universal path spaces and fundamental groups, these generalizations can be studied for topological groupoids.

R. Brown and G. Danesh-Naruie were the first and only ones that have defined a topology on a quotient of the fundamental groupoid so that it became a topological groupoid when the given space X is locally path connected and semilocally simply connected space [4].

Pakdaman and Shahini [7] equipped the fundamental groupoid of a locally path connected space X with a topology, named Lasso topology in which it can be considered as a generalization of the topological fundamental group.

At first, we show that for a given space X and every $x \in X$, $\pi_1^l(X, x)$ is Hausdorff if $\pi_1^{sp}(X, x) = \{e_x\}$. Then we use this to prove that the topological fundamental groupoid of Hausdorff spaces is Hausdorff if their Spanier groups are trivial.

1.2. **preliminaries.** Throughout this paper, all spaces are connected and locally path connected. I denotes the closed unit interval $[0, 1]$. If $\alpha : I \longrightarrow X$ is a path from $x_0 = \alpha(0)$ to $x_1 = \alpha(1)$, then α^{-1} defined by $\alpha^{-1}(t) = \alpha(1 - t)$ is the inverse path of α from x_1 to x_0. For $x \in X$, c_x is the constant path at x.
If $\alpha, \beta : I \longrightarrow X$ are two paths with $\alpha(1) = \beta(0)$, then $\alpha * \beta$ denotes the usual concatenation of the two paths. Also, all homotopies between paths are relative to end points.

Definition 1.1. *If $\mathcal{U}$ is an open cover of X, consider the subgroup of $\pi_1(X, x)$ consisting of the homotopy classes of loops that can be represented by a product of the following type.*

$$\prod_{j=1}^{n} u_j v_j u_j^{-1},$$

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where the u_j's are arbitrary paths starting at the base point x and each v_j is a loop inside one of the neighborhoods $U_i \in \mathcal{U}$. This group is called the Spanier group with respect to $\mathcal{U}$*, denoted by $\pi(\mathcal{U}, x)$ [8, 5] and the intersection of all the Spanier groups with respect to open covers is called Spanier group of X and is denoted by $\pi_1^{sp}(X, x)$.*

Definition 1.2 ([6]). *A* groupoid *is a small category in which every arrow is invertible, i.e. a groupoid G over G_0 consists of a set of arrows G and a set of objects G_0, together with the following five structure maps:*
$S : G \longrightarrow G_0$, called the source map,
$T : G \longrightarrow G_0$, called the target map,
$1 : G_0 \longrightarrow G; x \mapsto 1_x$, called the unit map,
$i : G \longrightarrow G; a \longmapsto a^{-1}$, called the inverse map,
$m : G_2 \longrightarrow G; (a; b) \mapsto m(a; b) = ab$, called the composition map, where G_2 denotes the set of composable arrows: $G_2 = \{(a; b) \in G \times G | \ S(b) = T(a)\}$.

These maps satisfy the following conditions:
i) $S(ab) = S(a)$ and $T(ab) = T(b)$ for all $(a; b) \in G_2$,
ii) $a(bc) = (ab)c$ for all $a, b, c \in G$ such that $S(b) = T(a)$ and $S(c) = T(b)$,
iii) $S(1_x) = T(1_x) = x$ for all $x \in G_0$,
iv) $a1_{T(a)} = a$ and $1_{S(a)}a = a$ for all $a \in G$,
v) each $a \in G$ has a two-sided inverse a^{-1} such that $S(a^{-1}) = T(a)$, $T(a^{-1}) = S(a)$ and $aa^{-1} = 1_{S(a)}$; $a^{-1}a = 1_{T(a)}$.

Elements of G_0 are called objects of the groupoid G and elements of G are called arrows. The arrow 1_x corresponding to an object $x \in G_0$ is called the identity corresponding to x. We denote the set of arrows from x to y by $G(x, y)$ and, in particular, $G(x) := G(x, x)$ is called the object group (or vertex group) at x.

Fundamental groupoid of X is a category in which the set of morphisms contains homotopy classes of paths in X, denoted by πX and has the set X as its set of objects. We will use πX for the entire category. For any $x, y \in X$ the set $\pi X(x, y)$ is the set of homotopy classes of paths in X from x to y. Composition of morphisms $[\alpha], [\beta]$ is $[\alpha * \beta]$ and the identity in $\pi X(x, x)$ is the $e_x = [c_x]$. We can consider the object group at x, $\pi X(x)$, as the well-known fundamental group $\pi_1(X, x)$.

Definition 1.3. *A* topological groupoid *is a groupoid G together with topologies on G and G_0 such that the structure maps are continuous [6].*

Let $\mathcal{U}$ be an open cover of a given space X and for $x, y \in X$, let $[\alpha] \in \pi X(x, y)$. If V, $W \in \mathcal{U}$ are open neighborhoods of x, y, respectively. Then

$$N([\alpha], \mathcal{U}, V, W) := \Big\{ [\beta] \in \pi X \mid \ \beta \simeq \gamma * \mu * \alpha * \mu' * \lambda, \text{ for som}\gamma : I \to V, \lambda : I \to W , [\mu] \in \pi_1(\mathcal{U}, x), [\mu'] \in \pi_1(\mathcal{U}, y) \Big\}.$$

The family

$$\Big\{ N([\alpha], \mathcal{U}, V, W) \big| \, \mathcal{U} \text{ is an open cover of } X; V, W \in \mathcal{U}, \ [\alpha] \in \pi X(x, y), \ x \in V, \ y \in W \Big\}$$

form a basis for a topology on πX [7]. The topology that is generated by this basis, is called **Lasso topology**. For a given topological space X, by $\pi^l X$ we mean the fundamental groupoid πX equipped with the Lasso topology on the set of morphisms and original topology on X, as the object set.

Theorem 1.4 ([7]). *Let X be a locally path connected topological space. Then $\pi^l X$ is a topological groupoid.*

2. Main results

Lemma 2.1. *Let G be a topological group with e as the identity element. Then G is a Hausdorff space if and only if $\{e\}$ is closed in G.*

Proof. Let G be a Hausdorff space. Since every T_2 space is T_1, $\{e\}$ is closed. Conversely, let $\{e\}$ be closed in G. Consider the continuous map $f : G \times G \to G$ defined by $f(g, h) = gh^{-1}$. Since the diagnal subset $f^{-1}(e)$ of $G \times G$ is closed, G is Hausdorff. $\square$

Lemma 2.2. *Let X be a topological space and $x \in X$. If the fundamental group is endowed with the Lasso topology, then $\pi_1^{sp}(X, x) = \overline{\{e_x\}}$.*

Proof. Let $[\alpha] \in \pi_1^{sp}(X, x)$, $\mathcal{U}$ be an open cover of the topological space X and $O := N([\alpha],\mathcal{U})$ be a basic open neighborhood of $[\alpha]$. Since $[\alpha] \in \pi_1^{sp}(X, x)$, for every open cover $\mathcal{U}$ of X, $[\alpha]$, $[\alpha]^{-1} \in \pi(\mathcal{U}, x)$. Thus $[\alpha]*[\alpha]^{-1} = [\alpha*\alpha^{-1}] = [c_x] = e_x \in O$. Therefore $O \cap \{e_x\} \neq \emptyset$ and the intersection of every basic open neighborhood of $[\alpha]$ with $\{e_x\}$ is nonempty, i.e. $[\alpha] \in \overline{\{e_x\}}$.

Conversely, if $[\gamma] \in \overline{\{e_x\}}$, then for every basic open neighborhood $N([\gamma],\mathcal{U})$ of $[\gamma]$, $N([\gamma],\mathcal{U}) \cap \{e_x\} \neq \emptyset$. Thus $e_x \in N([\gamma],\mathcal{U})$ and for some $[\beta] \in \pi(\mathcal{U}, x)$ we have $C_x \simeq \gamma*\beta$ which implies that $e_x = [\gamma]*[\beta]$. Obviously γ is a loop based at x, thus for every open cover $\mathcal{U}$ of X, $[\gamma] = e_x * [\beta^{-1}] \in \pi(\mathcal{U}, x)$ and hence $[\gamma] \in \pi_1^{sp}(X, x)$. $\square$

Corollary 2.3. *For every space X and every $x \in X$, $\pi_1^l(X, x)$ is Hausdorff if $\pi_1^{sp}(X, x) = \{e_x\}$.*

Proof. By Lemma 2.2, we have $\overline{\{e_x\}} = \{e_x\}$, so $\{e_x\}$ is closed in $\pi_1^l(X, x)$. Then Lemma 2.1 follows that $\pi_1^l(X, x)$ is Hausdorff. $\square$

It seems that by the condition $\pi_1^{sp}(X, x) = \{e_x\}$ on the Hausdorff space X, it can be proved that the fundamental groupoid is also Hausdorff by the Lasso topology. We prove this claim in the next theorem.

Theorem 2.4. *Let X be a Hausdorff space and $x \in X$. If $\pi_1^{sp}(X, x) = \{e_x\}$, then $\pi^l X$ is Hausdorff.*

Proof. Let $[\alpha] \neq [\beta] \in \pi^l X$.

(i) If $\alpha(0) \neq \beta(0)$, then there are open neighborhoods U and V of $\alpha(0)$ and $\beta(0)$ respectively, such that $U \cap V = \emptyset$. Let $\mathcal{U}'$ be an open cover of the topological space X and $\mathcal{U} := \mathcal{U}' \cup \{U, V\}$. If W and W' are elements of $\mathcal{U}$ containing $\alpha(0)$ and $\beta(0)$ respectively, then $O := N([\alpha],\mathcal{U}, U, W)$ and $O' := N([\beta],\mathcal{U}, V, W')$ are basic open neighborhoods of $[\alpha]$ and $[\beta]$ respectively. We show that $O \cap O' = \emptyset$.

Let $[\lambda]$ and $[\gamma]$ be arbitrary elements of O and O' respectively , then $\lambda(0) \in U$ and $\gamma(0) \in V$. Since $U \cap V = \emptyset$, we have $[\lambda] \neq [\gamma]$ and hence $O \cap O' = \emptyset$. If $\alpha(1) \neq \beta(1)$, by a similar proof we have $O \cap O' = \emptyset$.

(ii) $\alpha(0) = \beta(0) = x$, $\alpha(1) = \beta(1) = y$, $x \neq y$.

Let $\mathcal{U}'$ be an arbitrary open cover of the topological space X, U and V be open neighborhoods in X containing x and y respectively, and U' and V' be elements of $\mathcal{U}'$ contaning x and y respectively. Let $U'' = U' \cap U$ and $V'' = V' \cap V$. Now $\mathcal{U} := \mathcal{U}' \cup \{U'', V''\}$ is an open cover of X such that $\mathcal{U} \preceq \mathcal{U}'$. Suppose that $O := N([\alpha], \mathcal{U}, U'', V'')$ and $O' := N([\beta], \mathcal{U}, U'', V'')$ are basic neighborhoods of $[\alpha]$ and $[\beta]$. we show that $O \cap O' = \emptyset$. By contradiction, suppose that $O \cap O' \neq \emptyset$, then $O = O'$, and hence $[\beta] \in O' = O$. Therefore $\beta \simeq \lambda * \mu * \alpha * \mu' * \lambda'$ where λ and λ' are paths in open neighborhoods U'' and V'' respectively, $[\mu] \in \pi_1(\mathcal{U}, \alpha(0))$ and $[\mu''] \in \pi_1(\mathcal{U}, \alpha(1))$. By the hypothesis $\lambda(0) = \beta(0) = \alpha(0) = x = \lambda(1)$ and $\lambda'(1) = \beta(1) = \alpha(1) = y = \lambda'(0)$. Thus λ and λ' are loops in U'' and V'' respectively, and we have $[\lambda * \mu] \in \pi(\mathcal{U}, x)$ and $[\mu'' * \lambda''] \in \pi_1(\mathcal{U}, y)$. Obviously

$$\beta * \alpha^{-1} \simeq \lambda * \mu * \alpha * \mu'' * \lambda'' * \alpha^{-1}.$$

Since $[\alpha * (\mu' * \lambda'') * \alpha^{-1}] \in \pi(\mathcal{U}, x)$, $[\beta * \alpha^{-1}] \in \pi(\mathcal{U}, x) \leq \pi_1(\mathcal{U}', x)$. But $\mathcal{U}'$ is an arbitrary open cover of X which implies that $[\beta * \alpha^{-1}] \in \pi_1^{sp}(X, x)$. Therefore $[\beta * \alpha^{-1}] = e_x$ i.e. $\beta \simeq \alpha$, which is a contradiction. Thus $O \cap O' = \emptyset$.

(iii) if $\alpha(0) = \beta(0) = x$ and $\alpha(1) = \beta(1) = x$, then $[\alpha] \neq [\beta] \in \pi^l X(x, x)$. The proof is obvious by propositions 2.3. □

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THE LINEAR TOPOLOGY IN ABELIAN GROUPS

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Abstract. The notions of linear topology and the completion process in abelian groups are recalled. In particular, we discuss some special topologies defined over abelian groups with various set of neighborhoods about 0.

Key words and phrases: topology; linear; p-adic.

1. Introduction

Throughout this talk all groups are abelian with addition as a group operation. In the theory of abelian groups, topologies can be introduced in various ways which are natural in one sense or another. The importance of certain topologies will be evident from subsequent developments, especially when completeness will be discussed. In this talk the notions of linear topology and the completion process in abelian groups are recalled. In particular, we discuss some special topologies defined over abelian groups with various set of neighborhoods about 0. A good reference about the undefined notions is [1].

2. Main results

In this section we give the main notions and results concerning the linear topologies defined over abelian groups. Given an abelian group A, the first Ulm subgroup of A is denoted by A^1 and is defined as $A^1 = \cap_{n\in\mathbb{N}} nA$.

Definition 2.1. *Let A be an abelian group. The set of all subgroups of A is partially ordered under the inclusion relation. It is a lattice, where $B \cap C$ and $B + C$ are the lattice operations "inf" and "sup," respectively, for subgroups B and C of A. This lattice $\mathbf{L}(A)$ has a minimum and maximum element;* O *and A.*

Definition 2.2. *By a filtration of a set X of cardinality κ we mean a family $\{X_\alpha\}_{\alpha\leq\kappa}$ of subsets of X such that the following holds.*

(1) $\alpha \leq \beta$ *implies that* $X_\alpha \subseteq X_\beta$;
(2) $| X_\alpha |< \kappa$ *for all* $\alpha < \kappa$;
(3) $X_\beta = \cup_{\alpha<\beta} X_\alpha$ *whenever β is a limit ordinal;*
(4) $X = \cup_{\alpha<\kappa} X_\alpha$.

Definition 2.3. *A filter $\mathcal{D}$ on a set X is a set of subsets of X such that of subsets of X such that the following holds.*

(1) $\emptyset \notin \mathcal{D},\ X \in \mathcal{D}$;
(2) *if* $Y \in \mathcal{D}$ *and* $Y \subseteq Z \subseteq X$, *then* $Z \in \mathcal{D}$; *and*
(3) $U, V \in \mathcal{D}$ *implies* $U \cap V \in \mathcal{D}$.

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The most important topology on abelian groups to be considered are the linear topologies in which there is a base (fundamental system) of neighborhoods about 0 which consists of subgroups such that all the cosets of these subgroups form a base of open sets for the topology. A more formal definition of linear topologies cab be given as follows.

Definition 2.4. *Let $\mathbf{u}$ be a filter in the lattice $\mathbf{L}(A)$ of all subgroups of A. $\mathbf{u}$ defines a topology on A, if we declare the set of subgroups $U in \mathbf{u}$ to be a base of open neighborhoods about 0, and for every $a \in A$, the cosets $a + U (U \in \mathbf{u})$ as a base of open neighborhoods of a. We observe that all open sets will be unions of cosets $a + U$ with $a \in A$ and $U \in \mathbf{u}$. The continuity of the group operations is obvious. Thus A is always a group operation under the arising topology, which may be called the $\mathbf{u}$-topology of A; $(A, \mathbf{u})$ will denote A as a topological group equipped with the $\mathbf{u}$-topology.*

The following simple facts on **u**-topologies are deduced easily.

Proposition 2.5. *Let A be an abelian group. Then we have the following statements.*

(1) *The $\mathbf{u}$-topology on A is discrete exactly if $\{0\} \in \mathbf{u}$, indiscrete if $\mathbf{u} = \mathbf{A}$, Hausdorff if and only if $\cap_{U \in \mathbf{u}} U = 0$.*
(2) *Open subgroups are closed.*
(3) *If the $\mathbf{u}$-topology of A is Hausdorff, then it makes A into a 0-dimensional topological group.*
(4) *The closure of a subgroup B of A in the $\mathbf{u}$-topology of A is given by the formula $\bar{B} = \cap_{U \in \mathbf{u}} (B + U)$.*

Proof. See [1, Page 36]. ☐

The following special topologies are significant.

Proposition 2.6. *Let A be an abelian group. Then we have the following statements.*

(1) *The $\mathbb{Z}$-adic topology on a group A is defined by letting $\{nA : n \in \mathbb{N}\}$ be a base of neighborhoods about 0. This is a $\mathbf{u}$-topology, where $\mathbf{u}$ consists of all $U \leq A$ such that A/U is a bounded group. This topology is Hausdorff if and only if the first Ulm subgroup A^1 vanishes. A subgroup G of A is closed exactly if the first Ulm subgroup of A/G is 0.*
(2) *In the p-adic topology (for a prime p) the subgroups $p^k A (k < \omega)$ are declared to form a base of neighborhoods about 0. This is likewise a $\mathbf{u}$-topology, with $\mathbf{u}$ consisting of all $U \leq A$ such that A/U is a bounded p-group.*
(3) *In order to define the $\ddot{P}$rufer topology, we choose the filter $\mathbf{u}$ consist of all $U \leq A$ such that A/U satisfies the minimum condition on subgroups. This a Hausdorff topology in which all subgroups are closed.*
(4) *In the finite index topology, the subgroups of finite indices constitute a base of neighborhoods about 0; equivalently, $\mathbf{u}$ consists of the subgroups of finite indices in A. This is coarser than both the $\mathbb{Z}$-adic and the $\ddot{P}$rufer topologies.*

Proof. See [1, Page 37]. ☐

Example 2.7. *The multiplication group $\mathbb{T}$ of complex numbers of absolute value 1, and the additive group $\mathbb{R}$ of reals are usually viewed as being equipped with the interval topology (which is not a linear topology).*

We need the following definitions.

Definition 2.8. *Assume that a linear topology is defined on the group A in terms of a filter $\mathfrak{u}$ in the lattice $\mathbf{L}(A)$ of all subgroups of A. The subgroups $U \in \mathfrak{u}$ form a base of open neighborhoods about 0; we label them by a directed index set I, so that $i \leq j$ for for $i, j \in I$ means that $U_j \leq U_i$. Thus, I as a (directed) poset is dual-isomorphic to a subset of $\mathbf{u}$ (which has the natural order relation by inclusion).*

Definition 2.9. *By a net in A we mean a set $\{a_i\}_{i\in I}$ of elements in A, indexed always by I. A net is said to converge to a limit $a \in A$ if to every $i \in I$ there is a $j \in I$ such that $a_k - a \in U_i$ for all $k \geq j$.*

Definition 2.10. *A net $\{a_i\}_{i\in I}$ is a Cauchy net if to any given $i \in I$, there is a $j \in I$ such that $a_k - a_l \in U_i$ for all $k, l \geq j$.*

Definition 2.11. *A net $\{a_i\}_{i\in I}$ is neat if for every $i \in I$, the relation $a_k - a_i \in U_i$ for all $k \geq i$ (i.e., $j = i$ can be chosen).*

Definition 2.12. *Let A be an abelian group. Then A is said to be complete in a topology if it is Hausdorff, and every (neat) Cauchy net in A has a limit in A.*

Definition 2.13. *Assume $\{A_i : i \in I\}$ is a collection of groups, indexed by a poset I, and for each pair $i, j \in I$ of indices with $i \leq j$ there is given a connecting homomorphism $\pi_i^j : A_j \to A_i$ such that*

(1) *π_i^i is the identity map of A_i for all $i \in I$; and*
(2) *$i \leq j \leq k$ in I, then $\pi_i^j \pi_j^k = \pi_i^k$.*

In this case $\mathfrak{u} = \{A_i(i \in I); \pi_i^j\}$ is called an inverse system. By the (inverse) limit of this system is meant a group A^* such that

(1) *there are maps $\pi_i : A^* \to A_i$ such that $\pi_i = \pi_i^j \pi_j$ for all $i \leq j$; and*
(2) *If G is any group with maps $\rho_i : G \to A_i (i \in I)$ subject to $\rho_i = \rho_i^j \rho_j$ for all $i \leq j$, then is a unique map $\phi : G \to A^*$ satisfying $\rho_i = \pi_i \phi$ for all $i \in I$.*

There are two important completion processes in the completion of groups in the linear topology; Cauchy nets and inverse limits. The second method fits better in linear topologies.

Proposition 2.14. *Let A be a group with linear topology (not necessarily Hausdorff), and $\{U_i : i \in I\}$ a base of neighborhoods about 0, with I a directed index set: $i \leq j$ in I if and only if $U_i \geq U_j$. Define the groups $C_i = A/U_i$, and for $j \geq i$ in I, and the homomorphisms $\pi_i^j : C_j \to C_i$ via $\pi_i^j : a + U_j \to a + U_i$. The limit of the arising inverse system $\mathfrak{C} = \{C_i(i \in I) : \pi_i^j\}$ will be denoted by $\check{A}$: it is furnished with the topology inherited from the product topology of $\prod C_i$. Thus, if π_i denotes the ith projection $\prod C_i \to C_i$, then a subbase of neighborhoods about 0 in $\check{A}$ is given by the subgroups $\check{U}_i = \check{A} \cap \pi_i^{-1} 0$. Evidently, $\theta_A : a \longmapsto (\ldots, a + U_i, \ldots) \in \check{A}$ is a homomorphism which is continuous and open, and $\theta_A U_i = \theta_A A \cap \check{U}$ holds for each $i \in I$. It is clear that $\ker \theta_A$ is the intersection of all U_i.*

Proof. See [1, Page 37]. □

Lemma 2.15. *For every group with a linear topology, the group $\check{A}$ is complete in the induced topology.*

Proof. See [1, Lemma 7.5.]. □

We observe that the completion is always Hausdorff, and $\theta_A : A \to \breve{A}$ is monic if and only if A had a Hausdorff topology to start with.

Lemma 2.16. *If ϕ is a continuous homomorphism of the group A into a complete group C, then there is a unique continuous homomorphism $\breve{\phi} : \breve{A} \to C$ such that $\breve{\phi}\theta_A = \phi$.*

Proof. See [1, Lemma 7.6.]. □

From this lemma it also follows that the completion $\breve{A}$ of A is unique up to topological isomorphism. Moreover, $\theta_A : A \to \breve{A}$ is a natural map.

Theorem 2.17. *Let A be any group.*

(1) Its completion in the $\mathbb{Z}$-adic (p-adic) topology carries the $\mathbb{Z}$-adic (p-adic) topology.
(2) Its completion in the finite index topology has a compact topology.
(3) Its completion in the Prüfer topology carries a linearly compact topology.

Proof. See [1, Theorem 7.7]. □

Example 2.18. *Let p be a prime, and $\mathbb{Z}_{(p)}$ the ring of rational numbers whose denominators are prime to p. The non-zero ideals in $\mathbb{Z}_{(p)}$ are principal ideals generated by p^k with $k = 0, 1, \ldots.$ If the set of these ideals $p^k\mathbb{Z}_{(p)}$ is declared to be fundamental system of neighborhoods about 0, then $\mathbb{Z}_{(p)}$ becomes a (Hausdorff) topological ring, and we may form its completion J_p in this topology. We observe that J_p is a ring, called the ring of p-adic integers, whose non-zero ideals p^kJ_p with $k = 0, 1, \ldots,$ and which is complete in the topology in the topology defined by its ideals. The symbol J_p denotes both the ring and the group of p-adic integers. Moreover, $\mathbb{Q}_p^*$ denotes the field of quotients of J_p (and its additive group).*

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THE ORDER OF SAMELSON PRODUCT $Q_m \wedge Q_n \to Sp(n)$

SAJJAD MOHAMMADI

Abstract. Let m and n be two positive integers such that $m < n$. Let Q_n be the symplectic quasi-projective space of rank n. In this article, localise at an odd prime p, we will study the order of the Samelson product $Q_m \wedge Q_n \to Sp(n)$.

Key words and phrases: Samelson product; Symplectic group.

1. Introduction

Let H be a topological group. The commutator of H is the map $C : H \times H \to H$ defined by sending (a, b) to $aba^{-1}b^{-1}$. This is trivial when restricted to $H \vee H$ so induces a map $c : H \wedge H \to H$. The Samelson product of two maps $\alpha : X \to H$ and $\beta : Y \to H$ denoted by $\langle \alpha, \beta \rangle$ is defined to be the composition

$$X \wedge Y \xrightarrow{\alpha\wedge\beta} H \wedge H \xrightarrow{c} H.$$

Let G be a simple compact connected Lie group. The calculation of Samelson products plays an important role in classifying the homotopy types of gauge groups of principal G-bundles, and they are fundamental in studying the homotopy commutativity of Lie groups. Samelson products have been studied extensively for the classical groups. Let Q_m be the symplectic quasi-projective space of rank m, we denote the inclusion $Q_m \to Sp(n)$ by $\varepsilon_{m,n}$, where $m \leq n$. Let m and n two integers such that $m < n$. In this article, we will study the order of the Samelson product $\langle \varepsilon_{m,n}, \varepsilon_{n,n} \rangle : Q_m \wedge Q_n \to Sp(n)$. Let d be the order of the Samelson product $\langle \varepsilon_{m,n}, \varepsilon_{n,n} \rangle$.

Theorem 1.1. *Localise at an odd prime p. If $m = 2$ then d is $3^2 \cdot 7$ if $n = 3$ and is $3 \cdot 5 \cdot 11$ if $n = 4$.*

2. Preliminaries and Notations

Let X be a CW-complex. We denote $Sp(\infty)/Sp(n)$ by X_n and $[X, Sp(n)]$ by $Sp_n(X)$. Recall [2], that the symplectic quasi projective spaces Q_n of rank n for $n \leq 3$, have the following cellular structures

$$Q_1 = S^3, \quad Q_2 = S^3 \cup_{v_1} e^7, \quad Q_3 = S^3 \cup_{v_1} e^7 \cup_{v_2} e^{11},$$

where v_1 is a generator of $\pi_6(S^3)$ and $v_2 : S^{10} \to Q_2$ is some map. There is an inclusion $\varepsilon_{n,n} : Q_n \to Sp(n)$ that induces an isomorphism in homology $\Lambda(\widetilde{H}_*(Q_n)) \cong H_*(Sp(n))$. Consider the fibre sequence

$$\Omega Sp(\infty) \xrightarrow{\Omega\pi} \Omega X_n \xrightarrow{\delta} Sp(n) \xrightarrow{j} Sp(\infty) \xrightarrow{\pi} X_n, \tag{2.1}$$

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where $\pi : Sp(\infty) \to X_n$ is the projection map. Applying the functor $[X, -]$ to fibration (2.1), we get the following exact sequence.

$$\widetilde{KSp}^{-2}(X) \xrightarrow{(\Omega\pi)_*} [X, \Omega X_n] \xrightarrow{\delta_*} Sp_n(X) \xrightarrow{j_*} \widetilde{KSp}^{-1}(X) \xrightarrow{\pi_*} \tag{2.2}$$

Let γ be the commutator map $Sp(n) \wedge Sp(n) \to Sp(n)$. Since $Sp(\infty)$ is an infinite loop space it is homotopy commutative. Therefore γ composed to $Sp(\infty)$ is null homotopic, implying that there is a lift $\tilde{\gamma} : Sp(n) \wedge Sp(n) \to \Omega X_n$ such that $\delta \circ \tilde{\gamma} \simeq \gamma$. Therefore we get the following relation

$$\langle \varepsilon_{m,n}, \varepsilon_{n,n} \rangle = \delta_*(\tilde{\gamma} \circ (\varepsilon_{m,n} \wedge \varepsilon_{n,n})). \tag{2.3}$$

We denote the equivalence class of $\tilde{\gamma} \circ (\varepsilon_{m,n} \wedge \varepsilon_{n,n})$ in the cokernel of $(\Omega\pi)_*$ by $[\tilde{\gamma} \circ (\varepsilon_{m,n} \wedge \varepsilon_{n,n})]$ and the order of an element x of a group by $|x|$. Then by exact sequence (2.2) and relation (2.3), we have

$$|\langle \varepsilon_{m,n}, \varepsilon_{n,n} \rangle| = |[\tilde{\gamma} \circ (\varepsilon_{m,n} \wedge \varepsilon_{n,n})]|.$$

Therefore we will calculate $|[\tilde{\gamma} \circ (\varepsilon_{m,n} \wedge \varepsilon_{n,n})]|$.
We use the same symbol c' for the inclusion $Sp(n) \hookrightarrow U(2n) \hookrightarrow U(2n+1)$, the complexifications $BSp(n) \to BU(2n+1)$ and $BSp(\infty) \to BU(\infty)$. Let $\rho : [X, \Omega Y] \cong [\Sigma X, Y]$ be the adjoint. The following lemma has important role in the calculation of the order of the Samelson products in $Sp(n)$ [1].

Lemma 2.1. *Let X be a space. For a map $\alpha : \Sigma^2 X \to BSp(\infty)$,*

$$(\Omega\pi \circ \rho^2\alpha)^*(b_{4n+4k-2}) = (-1)^{n+k}(2n+2k-1)!\Sigma^{-2}ch_{2n+2k}(c'(\alpha)),$$

where Σ is the suspension isomorphism and ch_i is the $2i$ Chern character.

3. The Samelson product $\langle \varepsilon_{2,n}, \varepsilon_{n,n} \rangle$

Put $X = Q_2 \wedge Q_n$. Define a map

$$\lambda : [X, \Omega X_n] \to H^{4n+2}(X) \oplus H^{4n+6}(X) \cong \bigoplus_{1 \le i \le 3} \mathbb{Z}_i,$$

by $\lambda(\alpha) = (\alpha^*(b_{4n+2}), \alpha^*(b_{4n+6}))$, where $\mathbb{Z}_i = \mathbb{Z}$, $\alpha \in [X, \Omega X_n]$, b_{4n+2} and b_{4n+6} are generators of $H^{4n+2}(\Omega X_n) \cong H^{4n+6}(\Omega X_n) \cong \mathbb{Z}$. Note that λ is a homomorphism of groups that is monic. We denote the free abelian group with a basis $e_1, e_2, \ldots$, by $\mathbb{Z}\{e_1, e_2, \ldots\}$. Let $\alpha_1 : \Sigma Q_2 \to BSp(\infty)$ be the adjoint of the composition of the inclusions $Q_2 \to Sp(2) \to Sp(\infty)$ and $\alpha_2 : \Sigma Q_2 \to BSp(\infty)$ be the pinch map of the bottom cell $q : \Sigma Q_2 \to S^8$ followed by a generator of $\pi_8(BSp(\infty)) \cong \mathbb{Z}$. Note that $\widetilde{KSp}(\Sigma Q_2)$ is a free abelian group with a basis α_1, α_2, that is $\widetilde{KSp}(\Sigma Q_2) = \mathbb{Z}\{\alpha_1, \alpha_2\}$. Then we have$ch(c'(\alpha_1)) = \Sigma y_3 - \frac{1}{6}\Sigma y_7$ and $ch(c'(\alpha_2)) = -2\Sigma y_7$. Let $t_n : Q_n \to \Sigma\mathbb{C}P^{2n-1}$ be the restriction of $c' : Sp(n) \to SU(2n)$ to their quasi-projective spaces and $\eta \in \widetilde{K}(\mathbb{C}P^{2n-1})$ be the Hopf bundle minus the trivial line bundle. Put $\beta_{n,j} = t_n{}^*(\Sigma^2\eta^j)$, where $j = 1, 3, \ldots, 2n-1$. By [3], we have the following lemma.

Lemma 3.1. $\widetilde{K}(\Sigma Q_n) = \mathbb{Z}\{\beta_{n,j}, j = 1, 3, \ldots, 2n-1\}$.

Put $\theta_{i,n,j} = \mathbf{q}(c'(\alpha_i) \wedge \beta_{n,j}) \in \widetilde{KSp}(\Sigma Q_2 \wedge \Sigma Q_n)$, where $\mathbf{q} : K \to KSp$ is the quaternionization. We denote $\Sigma u_a \otimes \Sigma u_b$ by $u_{a,b}$.

Lemma 3.2. $\widetilde{KSp}^{-2}(X) = \mathbb{Z}\{\theta_{i,n,j}, i = 1, 2, \quad j = 1, 3, \ldots, 2n-1\}$.

3.1. **The case n=3.** The Chern characters of $\theta_{i,3,j}$, are given as

$$\begin{aligned}
ch(c'(\theta_{1,3,1})) &= -2u_{3,3} + \frac{1}{3}u_{3,7} - \frac{1}{60}u_{3,11} + \frac{1}{3}u_{7,3} - \frac{1}{18}u_{7,7} + \frac{1}{360}u_{7,11},\\
ch(c'(\theta_{1,3,3})) &= 2u_{3,7} - \frac{5}{2}u_{3,11} - \frac{1}{3}u_{7,7} + \frac{5}{12}u_{7,11},\\
&\vdots\\
ch(c'(\theta_{2,3,5})) &= 4u_{7,11}.
\end{aligned}$$

Lemma 3.3. *For $i = 1, 2$ and $j = 1, 3, 5$, Im $\lambda \circ (\Omega\pi)_*$ is generated by $\alpha_{i,3,j}$, where*

$$\begin{aligned}
\alpha_{1,3,1} &= -(\frac{2\cdot 7!}{5!}, \frac{7!}{3\cdot 3!}, \frac{9!}{3\cdot 5!}), \quad \alpha_{2,3,1} = -(0, \frac{2\cdot 7!}{3}, \frac{9!}{30}),\\
\alpha_{1,3,3} &= -(\frac{5\cdot 7!}{2}, \frac{7!}{3}, \frac{5\cdot 9!}{12}), \quad \alpha_{2,3,3} = -(0, 4\cdot 7!, 5\cdot 9!),\\
\alpha_{1,3,5} &= -(2\cdot 7!, 0, \frac{9!}{3}), \quad \alpha_{2,3,5} = (0, 0, -4\cdot 9!).
\end{aligned}$$

Proof. According to the definition the map of λ, we have

$$\lambda \circ (\Omega\pi)_*(\theta_{1,3,1}) = ((\Omega\pi \circ \rho^2\theta_{1,3,1})^*(b_{14}), (\Omega\pi \circ \rho^2\theta_{1,3,1})^*(b_{18})),$$

By Lemma 2.1, the calculation of the components is as follows:

$$\begin{aligned}
(\Omega\pi \circ \rho^2\theta_{1,3,1})^*(b_{14}) &= 7!\Sigma^{-2}ch_8(c'(\theta_{1,3,1})) = 7!\Sigma^{-2}(-\frac{1}{60}u_{3,11} - \frac{1}{18}u_{7,7}),\\
(\Omega\pi \circ \rho^2\theta_{1,3,1})^*(b_{18}) &= -9!\Sigma^{-2}ch_{10}(c'(\theta_{1,3,1})) = -9!\Sigma^{-2}(\frac{1}{360}u_{7,11}).
\end{aligned}$$

Therefore $\alpha_{1,3,1} = \lambda \circ (\Omega\pi)_*(\theta_{1,3,1}) = -(\frac{2\cdot 7!}{5!}, \frac{7!}{3\cdot 3!}, \frac{9!}{3\cdot 5!})$. Similarly we obtain other generators $\alpha_{i,3,j}$. □

Proposition 3.4. $|[\tilde{\gamma} \circ (\varepsilon_{2,3} \wedge \varepsilon_{3,3})]| = 3^2 \cdot 7$.

3.2. **The case n=4.** The Chern characters of $\theta_{i,4,j}$ are given as

$$\begin{aligned}
ch(c'(\theta_{1,4,1})) &= ch(c'(\theta_{1,3,1})) + \frac{2}{7!}u_{3,15} - \frac{1}{3\cdot 7!}u_{7,15},\\
ch(c'(\theta_{1,4,3})) &= ch(c'(\theta_{1,3,3})) + \frac{43}{60}u_{3,15} - \frac{43}{3\cdot 5!}u_{7,15},\\
&\vdots\\
ch(c'(\theta_{2,4,7})) &= -4u_{7,15}.
\end{aligned}$$

Lemma 3.5. *for $i = 1, 2$ and $j = 1, 3, 5, 7$, Im $\lambda \circ (\Omega\pi)_*$ is generated by $\alpha_{i,4,j}$, where*

$$\begin{aligned}
\alpha_{1,4,1} &= -(\frac{2\cdot 9!}{7!}, \frac{9!}{3\cdot 5!}, \frac{11!}{3\cdot 7!}), \quad \alpha_{2,4,1} = -(0, \frac{9!}{30}, \frac{4\cdot 11!}{7!}),\\
\alpha_{1,4,3} &= -(\frac{43\cdot 9!}{60}, \frac{5\cdot 9!}{12}, \frac{43\cdot 11!}{3\cdot 5!}), \quad \alpha_{2,4,3} = -(0, 5\cdot 9!, \frac{43\cdot 11!}{30}),\\
\alpha_{1,4,5} &= -(\frac{20\cdot 9!}{3}, \frac{9!}{3}, \frac{10\cdot 11!}{9}), \quad \alpha_{2,4,5} = -(0, 4\cdot 9!, \frac{40\cdot 11!}{3}),\\
\alpha_{1,4,7} &= -(2\cdot 9!, 0, \frac{11!}{3}), \quad \alpha_{2,4,7} = (0, 0, -4\cdot 11!).
\end{aligned}$$

Arguing as for Proposition 3.4 we obtain the following proposition.

Proposition 3.6. $|[\tilde{\gamma} \circ (\varepsilon_{2,4} \wedge \varepsilon_{4,4})]| = 3 \cdot 5 \cdot 11.$

Proof of Theorems 1.1
By Propositions 3.4, and 3.6 we get the Theorem 1.1.

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THE SELF HOMOTOPY GROUPS OF $SU(3)$

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Abstract. In this article, localized at odd prime 3, we will study the self-homotopy group of $SU(3)$.

Key words and phrases: Self homotopy group; $SU(3)$.

1. Introduction

Let G be a group-like space and X a finite complex. The homotopy set $[X, G]$ has a natural group structure inherited from G by the point-wise multiplication. In recent years, the group $[X, G]$ has been studied and there are many applications in homotopy theory, special in the homotopy commutativity and homotopy nilpotency. For a compact connected Lie group G, we call a group of the self homotopy set $[G, G]$ the self-homotopy group and is an interesting object in homotopy theory. The self-homotopy group of G has been studied extensively. In recent years, the self-homotopy groups of Lie groups are studied by topologists and good results have been obtained [1, 4, 5]. In [3], Kishimoto, Kono and Tsutaya study the self-homotopy group of $Sp(3)$ localized at $p \geq 5$. The purpose of this article is to study the self-homotopy groups of $SU(3)$ localized at primes 3. In this article, we denote by $-_{(p)}$ the localization at a prime p in the sense of Bousfield and Kan [2].

2. Preliminaries and Notations

In this section, we will study Unstable K-theory where all of spaces localized at odd prime 3. For a prime 3, we denote the 3-localization of a nilpotent group G by $G_{(3)}$. Also we denote the 3-localization of spaces by the same notation. Note that, for a CW-complex X, $[X, U(3)]_{(3)} = [X, U(3)_{(3)}]$. Let X be a CW-complex such that $\dim X \leq 8$. Also, let $H^*(X, \mathbb{Z})$ be a free $\mathbb{Z}_{(3)}$-module. We denote the infinite Stiefel manifold $U(\infty)/U(3)$ by W_3. Let $p\colon U(\infty) \to W_3$ is the projection map. By applying $[X,]$ to the fibration sequence

$$\Omega U(\infty) \xrightarrow{\Omega p} \Omega W_3 \xrightarrow{\delta} U(3) \xrightarrow{j} U(\infty) \xrightarrow{p} W_3,$$

we obtain the exact sequence

$$[X, \Omega U(\infty)] \xrightarrow{(\Omega p)_*} [X, \Omega W_3] \xrightarrow{\delta_*} [X, U(3)] \xrightarrow{j_*} [X, U(\infty)] \xrightarrow{p_*} [X, W_3].$$

By the Bott map $\beta\colon BU(\infty) \xrightarrow{\simeq} \Omega U(\infty)$, we have natural isomorphisms

$$[X, \Omega U(\infty)] \cong [X, BU(\infty)] \cong \tilde{K}^0(X), \quad [X, U(\infty)] \cong \tilde{K}^1(X).$$

Thus we get the following exact sequence.

$$\tilde{K}^0(X) \xrightarrow{(\Omega p)_*\beta_*} [X, \Omega W_3] \xrightarrow{\delta_*} [X, U(3)] \xrightarrow{j_*} \tilde{K}^1(X) \xrightarrow{p_*} [X, W_3]. \tag{2.1}$$

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It is well known that the cohomologies of $BU(\infty)$ and $U(\infty)$ as algebras are given respectively by $H^*(BU(\infty)) = \mathbb{Z}[c_1, c_2, \ldots]$, $H^*(U(\infty)) = \Lambda(x_1, x_3, \ldots)$, where $x_{2i-1} = \sigma(c_i)$, also c_i is the universal i-th Chern class and σ is the cohomology suspension. Localized at odd prime 3, we rewrite exact sequence (2.1) as the following theorem (see [3]).

Theorem 2.1. *Localized at odd prime* 3*, there is an exact sequence of groups*

$$\tilde{K}^0(X)_{(3)} \xrightarrow{\psi} Im(\lambda) \xrightarrow{\bar{\delta}} [X, U(3)]_{(3)} \xrightarrow{(j_*)_{(3)}} \tilde{K}^1(X)_{(3)},$$

such that for $\zeta \in \tilde{K}^0(X)_{(3)}$ *we have* $\psi(\zeta) = 3!ch_3(\zeta) \oplus 4!ch_4(\zeta)$*, where* $ch_i(\zeta)$ *is the* $2i$*-th part of* $ch(\zeta)$*.*

Let $\alpha_1, \alpha_2 \in [X, U(3)]_{(3)}$. The commutator $[\alpha_1, \alpha_2]$ in the group $[X, U(3)]_{(3)}$ was described explicitly in [3].

Theorem 2.2. *The commutator* $[\alpha_1, \alpha_2] \in [X, U(3)]_{(3)}$ *is equal to*

$$\bar{\delta}\Big(\bigoplus_{k=3}^{4} \sum_{\substack{i+j=k-1,\\ 1\le i,j\le 3}} {\alpha_1}^*(x_{2i+1}) \cup {\alpha_2}^*(x_{2j+1})\Big).$$

Therefore by Theorems 2.1 and 2.2 we obtain the following corollary that is very important in the study of self homotopy groups.

Corollary 2.3. *There is a central exact sequence*

$$0 \to Coker\psi \to [X, U(3)]_{(p)} \to Im(j_*)_{(3)} \to 0.$$

3. Self-homotopy groups of $SU(3)$ at prime $p = 3$

In this section, we will study the Self-homotopy group $[SU(3), U(3)]$ at prime $p = 3$. We denote the free abelian group with a basis $e_1, e_2, \ldots$, by $\mathbb{Z}\{e_1, e_2, \ldots\}$. We know that there is a mod 3 decomposition of $SU(3)$ as following

$$SU(3) \simeq_{(3)} S^3 \times S^5.$$

For $i = 2, 3$, let $\pi_i\colon SU(3)_{(3)} \to S^{2i-1}_{(3)}$ and $l_i\colon S^{2i-1}_{(3)} \to SU(3)_{(3)}$ be the projection and inclusion maps, respectively. We define the self map $\xi_{3,i}$ of $SU(3)$, as the following composition

$$SU(3)_{(3)} \xrightarrow{\pi_i} S^{2i-1}_{(3)} \xrightarrow{l_i} SU(3)_{(3)}.$$

Now, put ${\xi'}_{4,i}$ and $\bar{\xi}_{4,i}$ as the following compositions

$$SU(3)_{(3)} \xrightarrow{\xi_{3,i}} SU(3)_{(3)} \to U(3)_{(3)},$$

$$SU(3)_{(3)} \xrightarrow{\xi_{3,i}} SU(3)_{(3)} \to U(3)_{(3)} \xrightarrow{j} U(\infty)_{(3)},$$

respectively. Note that for $i = 2, 3$, we have $\bar{\xi}_{3,i} \in \tilde{K}^1(SU(3))$ with the Chern character $ch(\bar{\xi}_{3,i}) = \frac{1}{(i-1)!}\Sigma x_{2i-1}$. By Corollary 2.3, there is a central exact sequence

$$0 \to Coker\psi \to [SU(3), U(3)]_{(3)} \to Im(j_*)_{(3)} \to 0.$$

So we need to calculate $Im(j_*)_{(3)}$ and $Coker\psi$. First, we compute the image of map

$$(j_*)_{(3)}\colon [SU(3), U(3)]_{(3)} \to \tilde{K}^1(SU(3))_{(3)}.$$

Note that $\tilde{K}^1(SU(3))_{(3)}$ is a free $\mathbb{Z}_{(3)}$-module generated by $\bar{\xi}_{3,2}$, $\bar{\xi}_{3,3}$. We have the following lemma.

Lemma 3.1. *$Im(j_*)$ is a free $\mathbb{Z}_{(3)}$-module generated by $\bar{\xi}_{3,2}$, $\bar{\xi}_{3,3}$.*

Now, we compute the Coker of the map $\psi\colon \tilde{K}^0(SU(3))_{(3)} \to Im\lambda$. Consider the map

$$\lambda\colon [SU(3), \Omega W_3]_{(3)} \to H^8(SU(3); \mathbb{Z}_{(3)}),$$

where $\lambda(\alpha) = \alpha^*(a_8)$. By [3], we know that the map λ is monic and $Im(\lambda)$ is generated by x_3x_5. Let $\vartheta = \beta^{-1}(\bar{\xi}_{3,2}\bar{\xi}_{3,3})$. Note that $\tilde{K}^0(SU(3))_{(3)}$ is a free $\mathbb{Z}_{(3)}$-module generated by ϑ, with the Chern character $ch(\vartheta) = x_3x_5$. According to the definition of map ψ, we have $\psi(\vartheta) = 4!ch_4(\vartheta) = 4!x_3x_5$. Thus localized at odd prime 3, $Im(\psi)$ is generated by $3x_3x_5$, therefore we get the following lemma.

Lemma 3.2. *There is an isomorphism*

$$Coker\psi \cong \frac{< x_3x_5 >}{< 3x_3x_5 >} \cong \mathbb{Z}_3.$$

By Theorem 2.2, we have that the commutator $[\xi'_{3,2}, \xi'_{3,3}]$ in the group $[SU(3), U(3)]_{(3)}$ is equal to $\bar{\delta}(x_3x_5)$. Therefore we get the following theorem.

Theorem 3.3. *There is a central exact sequence*

$$0 \to \mathbb{Z}_3 \to [SU(3), U(3)]_{(3)} \xrightarrow{f} \mathbb{Z}_{(3)} \oplus \mathbb{Z}_{(3)} \to 0,$$

such that the following hold.
(a) : $f(\xi'_{3,2}) = (1, 0)$ and $f(\xi'_{3,3}) = (0, 1)$, respectively.
(b) : $Kerf = \mathbb{Z}\{[\xi'_{3,2}, \xi'_{3,3}]\}$.

On the other hand, we know that $U(3) \simeq S^1 \times SU(3)$. Now, by homotopy groups of sphere we have that for $i = 3, 5$, $\pi_i(S^1)_{(3)} = 0$. Therefore there is an isomorphism $[SU(3), SU(3)]_{(3)} \cong [SU(3), U(3)]_{(3)}$. Thus we obtain the following theorem.

Theorem 3.4. *There is a central exact sequence*

$$0 \to \mathbb{Z}_3 \to [SU(3), SU(3)]_{(3)} \xrightarrow{f} \mathbb{Z}_{(3)} \oplus \mathbb{Z}_{(3)} \to 0,$$

such that the following hold.
(a) : $f(\xi'_{3,2}) = (1, 0)$ and $f(\xi'_{3,3}) = (0, 1)$, respectively.
(b) : $Kerf = \mathbb{Z}\{[\xi'_{3,2}, \xi'_{3,3}]\}$.

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POISSON-NIJENHUIS STRUCTURE ON CO-ADJOINT LIE GROUPOIDS

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Abstract. Our purpose in this article is to define the compatible Poisson - Nijenhuis structure of the Co-adjoint orbit $\mathcal{G}_\xi$ of a Lie groupoid G. That is, we consider a Poisson-Nijenhuis structure for a Lie groupoid G and by using it, we calculate the Poisson-Nijenhuis structure corresponding to its Co-adjoint Lie groupoid. In the end, considering the trivial Lie groupoid as an example, we obtain the Poisson-Nijenhuis structure for its corresponding Co-adjoint Lie groupoid.

Key words and phrases: Poisson-Nijenhuis groupoid; Co-adjoint Lie groupoid; Poisson-Nijenhuis structure.

1. Introduction

We know that the theory of Lie groupoids is extended by the theory of Lie groups. Also, in the second section of this article, we can recall the notion of a Lie algebroid associated with a Lie groupoid. In accord with [4], the notion of tangent Lie algebroid and cotangent Lie groupoid corresponding to a Lie groupoid is defined. In the following Das defined multiplicative Poisson-Nijenhuis structures on a Lie groupoid which extends the notion of symplectic-Nijenhuis groupoid expressed by Stienon and Xu in [1]. The notion of Poisson groupoid was introduced by Weinstein [5] as a union of both the Poisson Lie group and the symplectic groupoid. We can see that the concept of Poisson-Nijenhuis has been studied by Magri and Morosi. By referring to [2] we saw that the orbits of the Co-adjoint action of a Lie groupoid are obtained by an action of a Lie groupoid on a dual bundle of the isotropy Lie algebroid associated with isotropy Lie groupoid of a Lie groupoid. Likewise, there is a (1,1) - Nijenhuis tensor on a manifold that this manifold together with this (1,1) - Nijenhuis tensor on itself are a Nijenhuis manifold which these two are a compatible structure. In other words, it is satisfying in two compatible conditions. As well as, we can see that for a smooth manifold, there is a 1-vector-valued form or a (1,1) - tensor on a manifold that if it is with a Nijenhuis torsion zero, then we can call this 1-form vector-valued is a (1,1) - Nijenhuis tensor. Due to the assumption of the problem and the mentioned concepts in [1], we have that a Poisson-Nijenhuis Lie groupoid is include a Poisson groupoid which itself includes a Lie groupoid together with a Poisson bivector field for which the Schouten bracket is zero and also a Nijenhuis groupoid.

We only have enough to identify the manifold with the Lie groupoid and therefore we will obtain that the orbits of the Co-adjoint action of a Lie groupoid together with a Poisson-Nijenhuis structure are associated with Poisson-Nijenhuis structure

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on a manifold. Finally, we express an example of Lie groupoids in the name of trivial Lie groupoids [4] and also obtain the Co-adjoint orbit action of trivial Lie groupoids. Since that based on [2, 4] a trivial groupoid is a Lie groupoid.Then, we could define a Poisson-Nijenhuis structure for a trivial Lie groupoid. Thereby, we will show that a trivial Lie groupoid is a Poisson-Nijenhuis Lie groupoid in the last section.

2. Main results

Definition 2.1. *A groupoid which is denoted by $G \rightrightarrows M$, consists of two sets G and M together with structural mappings $s, t, 1, \iota$ and m, where source mapping $s : G \to M$, target mapping $t : G \longrightarrow M$, unit mapping $1 : M \longrightarrow G$, inverse mapping $\iota : G \longrightarrow G$ and multiplication mapping $m : G_2 \longrightarrow G$ which $G_2 = \{(g, h) \in G \times G \mid s(g) = t(h)\}$ is a subset of $G \times G$.*
A Lie groupoid is a groupoid $G \rightrightarrows M$ for which G and M are smooth manifolds, $s, t, 1, \iota$ and m, are differentiable mappings and besides, s, t are differentiable submersions .

Definition 2.2. *A Lie algebroid over a manifold M consists of a vector bundle A together with a bundle map $\rho_A : A \longrightarrow TM$ and a Lie bracket $[\ ,\ ]_A$ on the space of sections $\Gamma(A)$, satisfying the Leibniz identity*

$$[\alpha, f\beta]_A = f\,[\alpha, \beta]_A + L_{\rho_{A(\alpha)}}(f)\beta,$$

for $\alpha, \beta \in \Gamma(A)$ and all $f \in C^\infty(M)$.

Definition 2.3. *A vector field X on G is called vertical if it is vertical with respect to s, that is, $X_g \in T_g G_{s(g)}$, for all $g \in G$. We call X right - invariant on G if it is vertical and $X_{gh} = dR_g(X_h)$, for all $(h, g) \in G_{(2)}$.*
It is easy to show that $\Gamma(AG)$ - the space of sections of vector bundle AG can be identified as the space of the right - invariant field on G. So if we denote the space of right- invariant vector field on G by

$$\chi^s_{\mathrm{inv}}(G) = \Big\{X \in \Gamma(T^s G) : X_{hg} = dR_g(X_h),\ (h, g) \in G_{(2)}\Big\}.$$

From above we have the space of sections $\Gamma(AG)$ is isomorphic to the space of right - invariant vector fields on G, $\chi^s_{\mathrm{inv}}(G)$. On the other hand, the space $\chi^s_{\mathrm{inv}}(G)$ is a Lie sub-algebra of the Lie algebra $\chi(G)$ of a vector field on G concerning the usual Lie bracket of vector fields. Also, the pull-back of the vector field on the $s-$fibers along R_g preserves brackets. So we obtain a new bracket on $\Gamma(AG)$ which is uniquely determined. The Lie bracket on AG is the Lie bracket on $\Gamma(AG)$ obtained from the Lie bracket on $\chi^s_{\mathrm{inv}}(G)$. The anchor of AG is the differential of the target mapping β, i.e. $\rho = Tt|_{AG} : AG \to TM$. As a result, we obtain that AG is a Lie algebroid associated with the Lie groupoid G. Let X be a section of $\tau : AG = T^s G|_M \to M$, i.e. $X \in \Gamma(AG)$. We consider the right invariant vector field $\overrightarrow{X} : G \to TG, \overrightarrow{X}(g) = dR_g(X_{t(g)})$, where $dR_g : T^s_{1_{t(g)}} G \to T^s_g G$.

Definition 2.4. *Let $G \rightrightarrows M$ be a Lie groupoid. We define the orbit of the Co-adjoint action of a Lie groupoid G as follows:*

$$O(\xi) = \left\{Ad^*_g \xi | g \in G\right\}$$

where ξ is an element of $(A^* I_G)_p$. We call $O(\xi)$ Co-adjoint orbit of the Lie groupoid G.

Definition 2.5. *Let $G \rightrightarrows M$ be a Lie groupoid with a Poisson structure π on the Lie groupoid G. Then (G, π) is a Poisson groupoid i.e. the Poisson anchor $\pi^\sharp : T^*G \longrightarrow TG$ is a morphism of groupoids.*

Definition 2.6. *The Poisson structure of the Co-adjoint orbits of a Lie groupoid is as follows $\pi_\xi : T\mathcal{G}_\xi \longrightarrow T\mathcal{G}_\xi$ that $\pi_\xi(ad^*_X\xi) = ad^*_{\pi(X)}\xi$. where $\xi \in A^*I_G$ is normal.*

Definition 2.7. *Let π_ξ be a Poisson structure corresponding to Co-adjoint Lie groupoid $\mathcal{G}_\xi$. Then, we have*

$$\pi^\sharp_\xi : T^*\mathcal{G}_\xi \longrightarrow T\mathcal{G}_\xi \quad , \quad \pi^\sharp_\xi(\gamma) = ad^*_{\pi^\sharp(\alpha)}\xi.$$

Theorem 2.8. *Let $(G \rightrightarrows M, \pi)$ be a Poisson groupoid. Then $(\mathcal{G}_\xi \rightrightarrows M, \pi_\xi)$ is a Poisson groupoid.*

Definition 2.9. *Let N_M be a $(1, 1)$- Nijenhuis tensor on manifold M. Then, we can also define a $(1, 1)$- multiplicative Nijenhuis tensor on $\mathcal{G}_\xi$ in the form of N_ξ : $T\mathcal{G}_\xi \longrightarrow T\mathcal{G}_\xi$ for which $ad^*_X\xi \longmapsto ad^*_{N(X)}\xi$.*

Theorem 2.10. *Let $(G \rightrightarrows M, N)$ be a Nijenhuis groupoid. Then $\left(\mathcal{G}_\xi \rightrightarrows M, N_{_\xi}\right)$ is a Nijenhuis groupoid.*

Definition 2.11. *We can also define the dual of N_ξ in the form of $N^*_\xi : T^*\mathcal{G}_\xi \longrightarrow T^*\mathcal{G}_\xi$ in which $\gamma \longmapsto N^*_\xi(\gamma) := (N_\xi(ad^*_X\xi))^* = (ad^*_{N(X)}\xi)^*$.*

Theorem 2.12. *Let (π, N) be a Poisson-Nijenhuis structure on Lie groupoid. Then $(\pi_\xi, N_{_\xi})$ is a Poisson-Nijenhuis structure on Co-adjoint Lie groupoid.*

Theorem 2.13. *Let $(G \rightrightarrows M, \pi, N)$ be a Poisson - Nijenhuis Lie groupoid. Then $(\mathcal{G}_\xi, \pi_\xi, N_\xi)$ is a Poisson - Nijenhuis Lie groupoid.*

Proof. it is proved according to theorems (2.8,2.10,2.12). □

Example 2.14. *Let $\Upsilon := M \times G \times M$ be a trivial Lie groupoid for which M a Poisson-Nijenhuis manifold and G is a Poisson-Nijenhuis Lie group and $\mathfrak{g}$ is a Lie algebra of G and $\mathfrak{g}^*$ is a dual of Lie algebra $\mathfrak{g}$. As shown in article [3], $\Upsilon = M \times G \times M$ is a Poisson-Nijenhuis groupoid whose Poisson structure is $\pi_\Upsilon = \pi_M \oplus \pi_G \oplus \pi_M$ and Nijenhuis structure is $N_\Upsilon = N_M \oplus N_G \oplus N_M$, which π_M is the Poisson structure of M and π_G is also the Poisson structure of G.*
In addition, as stated in article [2], the Co-adjoint orbit of a trivial Lie groupoid is

$$\mathcal{G}_\xi = O(\xi) = \{Ad^*_h\xi | h \in \Upsilon\} = \{(q, Ad^*_b\xi', q) | b \in G, q \in M\},$$

for $\xi = (q, \xi', q) \in (A^*I_\Upsilon)_q, \xi' \in \mathfrak{g}^*$ that is normal. As well as,

$$\pi_\xi : T(M \times \mathcal{G}_{\xi'}) \longrightarrow T(M \times \mathcal{G}_{\xi'}), ad^*_{\tilde{X}}\xi \longmapsto \pi_\xi(ad^*_{\tilde{X}}\xi) := ad^*_{\pi_\Upsilon(X,V,Y)}\xi,$$

and

$$\pi^\sharp_\xi(\gamma') := ad^*_{\pi^\sharp_\Upsilon(\alpha')}\xi.$$

On the other hand,

$$N_\xi : T(M \times \mathcal{G}_{\xi'}) \longrightarrow T(M \times \mathcal{G}_{\xi'}), ad^*_{\tilde{X}}\xi \longmapsto N_\xi(ad^*_{\tilde{X}}\xi) := ad^*_{N_\Upsilon(X,V,Y)}\xi,$$

$$N^*_\xi : T^*(M \times \mathcal{G}_{\xi'}) \longrightarrow T^*(M \times \mathcal{G}_{\xi'}), \gamma' \longmapsto N^*_\xi(\gamma') := (N_\xi(ad^*_{\tilde{X}}\xi))^* = (ad^*_{N_\Upsilon(X,V,Y)}\xi)^*.$$

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THE LIBERATION PROCESS FROM QUANTUM PERMUTATION GROUPS TO QUANTUM (ALGEBRAIC) MANIFOLDS

FARROKH RAZAVINIA AND GHORBANALI HAGHIGHATDOOST BONAB

Abstract. The hyperoctahedral group H_n is known to have two natural liberations H_n^+ and $\overline{O}_N$. In this paper, we will study this phenomenon by using the framework introduced by Wang and Banica and we will present an almost independent proof of the fact that the quantum isometry group of K_n^+ (the n-dimensional quantum hypercube) is H_n^+ (the non-commutative hyperoctahedral group), and at the end, as a toy example, we will study the quantum isometry group of the d-dimensional lemon in $\mathbb{R}^d$ (as a classical compact space), and its maximal quantum version, and we show that the classical version has indeed some genuine quantum symmetry and as a result we conclude that the quantum isometry group $QISO^+(X)$ of any classical compact space X has to be classical.

Key words and phrases: liberation process; quantum permutation group; quantum isometry group.

1. Introduction

The structure of the usual sphere S^{n-1} is intimately related to that of the orthogonal group O_n, and when twisting the sphere the orthogonal group gets twisted as well, and becomes a quantum group. Quantum groups have been an object of study for many years now and especially the C^*-algebraic compact quantum groups (CQG) introduced by Woronowicz possess a very powerful representation theory. Actions of quantum groups on C^*-algebras dualize the idea of group actions and describe the symmetries of an object in the non-commutative case; one gets some kind of "quantum symmetry". Wang showed in [11] that even classical objects can have quantum symmetry unseen by restricting to classical groups. For example the set of n points gives rise to a commutative C^*-algebra but has genuine quantum symmetry. Its quantum symmetry group is the famous (compact) quantum group S_n^+ which is not a group for $n \geq 4$ and is infinite dimensional in the later cases. But allowing arbitrary quantum group actions on compact spaces often ignores too much information of the compact space; since we are working with C^*-algebras, only the topology of the space is taken into consideration. In that sense a square and a rectangle have the same quantum symmetry group. Alain Connes defined with the help of spectral triples, quantum group actions on Riemannian Manifolds that preserve the differential structure of the manifold. Generalizing this further, Goswami [6], Banica[1], Bichon and Collins have defined and studied isometric quantum group actions on classical finite and compact metric spaces discovering for example the

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non-commutative version of the hyperoctahedral group H_n^+ in search for the quantum isometry group of the n-dimensional hypercube in [4]. Rieffel also introduced the notion of a non-commutative metric space and Quaegebeur and Sabbe then defined isometric quantum group actions on such non-commutative metric spaces in [8]. It is not yet clear if the two notions of isometric actions introduced by Goswami and Quaegebeur-Sabbe respectively are equivalent on classical spaces. Besides what I said above, there is the idea of non-commuting coordinates which goes back to Heisenberg. Several theories emerged from Heisenberg's work, most complete being Connes' noncommutative geometry, where the base space is a Riemannian manifold. I this order, the algebra generated by variables u_{ij} with relations making $u = (u_{ij})$ a unitary matrix was considered by Brown. This algebra has a comultiplication and a counit, but no antipode (hence the corresponding non-commutative version U_n^{nc} was a quantum semigroup), hence they didn't fit into Woronowicz's axioms, but with a slight modification they could lead to free quantum groups. The quantum groups O_n^+, U_n^+ appeared in Wang's thesis [10]. Then Connes suggested use of symmetric groups, and the quantum group S_n^+ was constructed in [11].

The purpose of this paper is to study a new free quantum group which has been introduced in [4], and is the free analogue of the hyperoctahedral group H_n and is denoted by $H_n^+ \subseteq S_{2n}^+$ and satisfies in $S_n^+ \subset H_n^+ \subset U_n^+$. Let us recall that H_n is the common symmetry group of the cube $K_n \subset \mathbb{R}^n$ (considered as the metric space of 2^n points or the cubic graph with 2^n vertices, $n2^{n-1}$ edges) and the space $I_n \subset \mathbb{R}^n$ formed by the ± 1 points on each axis (also regarded as a graph formed by n segments, $2n$ vertices, n edges). According to the quantum algebra theory, the relations $ab = ba$ between the coordinates $x_1, \ldots, x_N$ of our ambient space $\mathbb{R}^N$ is $ab = -ba$ for $a \neq b$ and almost the same at the matrix level, else than the fact that we have $ab = -ba$ for $a \neq b$ on the same row or column of $u \in M_N(\mathbb{R})$ (let's call these new CCR relations, just $\mathfrak{R}$) and can be used in order to construct a twisted analogue of the orthogonal group O_N, as abstract spectrum of the universal algebra $C(\overline{O}_N) = C^*\left((u_{ij})_{i,j=1,\ldots,N} \mid u = \overline{u}, u^t = u^{-1}, \mathfrak{R}\right)$.
Generally speaking, the structure of $\overline{O}_N$ is quite similar to that of O_N, with the correspondence $O_N \rightleftharpoons \overline{O}_N$ being best understood via Schur-Weyl twisting, or via a cocycle deformation method. One interesting feature of $\overline{O}_N$, however, which escapes the philosophy of the above correspondence, is that this appears as quantum symmetry group of the standard hypercube in $\mathbb{R}^N$. This phenomenon was discovered about 10 years ago, in [4], and has been since the subject of various investigations. The following key construction is due to Wang [10].

Proposition 1.1. *We have a compact quantum group* O_n^+ *, defined as* $C(O_n^+) = C^*\left((u_{ij})_{i,j=1,\cdots,N} \mid u = \overline{u}, u^t = u^{-1}\right)$. *This quantum group contains* O_N*, and the inclusion* $O_N \subset O_N^+$ *is not an isomorphism.*

Both classically and in the quantum framework the simplest symmetry groups are (quantum) permutation groups, which can be viewed as the universal (quantum) groups acting on a given finite set as has been proved by Wang [11] as a result stating that the category $\mathfrak{C}(\mathbb{C}^n)$ of quantum groups acting on the n-point set admits a universal object which is denoted by S_n^+. But this statement is not true when dealing with the space of $n \times n$ matrices $M_n(\mathbb{R})$ and the category $\mathfrak{C}(M_n(\mathbb{R}))$ does not admit a universal object if $n > 1$ [11]. The problem is related to the fact that there is a universal object in the category of compact quantum semigroups acting on $M_n(\mathbb{R})$, but it is not a compact quantum group.

In infinite dimensional cases, as the last two statements above showed, in general to establish the existence of a quantum symmetry group of some C^*-algebra B we need to put some more structure on B. This has led to the development of the theory of quantum isometry groups of non-commutative manifolds initiated by Goswami and further developed by Banica, Bhowmick, and others.
Let Γ be a finitely generated discrete group with (minimal, symmetric) generating set $S = \{\gamma_1, \ldots, \gamma_n\}$ and $I : \Gamma \to \mathbb{N}_0$ be the word-length function.

Theorem 1.2 (J. Bhowmick)**.** *The category of all compact quantum groups acting on $C^*(\Gamma)$ and 'preserving the length' has a universal object; we call it the quantum isometry group of $\hat{\Gamma}$ and denote by $QISO^+(\hat{\Gamma})$. The claim is that $QISO^+(\hat{\Gamma})$ is a compact matrix quantum group with a fundamental representation $[q_{t,s}]_{t,s\in S}$, where the elements $\{q_{t,s} : t, s \in S\}$ must satisfy the commutation relations implying that the prescription $\alpha(\lambda_\gamma) = \sum_{\gamma' \in S : I(\gamma) = I(\gamma')} q_{\gamma',\gamma} \otimes \lambda_{\gamma'}$, for $\gamma \in \Gamma$ defines (inductively) a unital $*$-homomorphism from $C^*(\Gamma)$ to $C(QISO^+(\hat{\Gamma}) \otimes C^*(\Gamma))$.*

In almost recent years T.Banica, B.Collins, S.Curran, R.Speicher (and others) have initiated the study of a so-called liberation procedure. The idea can be (very informally) described as follows:

(1) consider your favourite compact group of matrices G,
(2) find a presentation of $C(G)$ in terms of finitely many generators, preferably coefficients of a unitary representation,
(3) 'liberate' the generators, that is drop the assumption that they must commute,
(4) show that the resulting family of algebraic relations determines an algebra $C(\mathbb{G})$ for a certain compact quantum group $\mathbb{G}$. We usually write $\mathbb{G} = G^+$,

and we have the following result concerning the liberations of H_n.

Proposition 1.3 (Banica)**.** *The hyperoctahedral group H_N has at least two natural liberations, namely $H_N \subset H_N^+$ and $H_N \subset \overline{O}_N$, and neither of them is universal.*

2. Is there a connection?

We have seen that the quantum permutation group S_n^+ can be viewed on one hand as the quantum symmetry group of the n-point set, and on the other as the liberation of the classical permutation group S_n. Now the question is that, are there any more examples of that type?

Theorem 2.1 (Banica, Bhowmick)**.** *For $\mathbb{F}_n$, the free group on n generators with the usual generating set, $QISO^+(\widehat{\mathbb{F}_n})$ can be described explicitly and turns out to be the liberation of the classical group $\mathbb{T}^n \rtimes H_n$.*

Remark 2.2. *The above Theorem might look somehow mysterious, but we also have $\mathbb{T}^n \rtimes H_n = ISO(\mathbb{T}^n) = ISO(\widehat{\mathbb{Z}^n})$.*

Which means that in fact here the object whose symmetry group we compute is being liberated! In a proccetrure, by using a mixture of the free probabilistic quantum group techniques we can compute the representation theory of $QISO^+(\widehat{\mathbb{F}_n})$, using the combinatorial language of (non-crossing) partitions, which in turn allows us to prove results of the following types.

Theorem 2.3 (Banica, Skalski). *The quantum group $QISO^+(\widehat{\mathbb{F}_n})$ is isomorphic to the quantum group H_n^+.*

Proposition 2.4 (Razavinia, Haghighatdoost). *The quantum isometry group $QISO^+(Y)$ of any classical quantum space Y is classical.*

3. Some open directions

(1) Geometric aspects. The groups S_n, O_n and their quantum (free) versions S_n^+, O_n^+ were involved in many other "classical vs. free" considerations. Let us mention here the Poisson boundary results in [9], and the quantum isometry groups in [7]. And it's good to mention that the easy quantum groups can lead to some new results here.

(2) Eigenvalue computations. The key results of Diaconis and Shahshahani in [5], concerning S_n, O_n has been obtained as well by using Weingarten functions and cumulants, and an extension to all easy quantum groups has been constructed and the original philosophy suggested in [3], namely the fact that "any result which holds for S_n, O_n should have an extension to easy quantum groups", has been illustrated. Now the question is that "What are the eigenvalues of a random quantum group matrix?".

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CONTACT PSEUDO-METRIC STRUCTURES ON TANGENT SPHERE BUNDLES

NARGES GHAFFARZADEH AND MORTEZA FAGHFOURI

Abstract. In this paper, we introduce a contact pseudo-metric structure on a tangent sphere bundle $T_\varepsilon M$. we prove that the tangent sphere bundle $T_\varepsilon M$ is (κ,μ)-contact pseudo-metric manifold if and only if the manifold M is of constant sectional curvature. Also, we prove that this structure on the tangent sphere bundle is K-contact iff the base manifold has constant curvature ε.

Key words and phrases: contact pseudo-metric structure, tangent sphere bundle, unit tangent sphere bundle, Sasaki pseudo-metric.

1. Introduction

In 1956, S. Sasaki [7] introduced a Riemannian metric on tangent bundle TM and tangent sphere bundle T_1M over a Riemannian manifold M. Thereafter, that metric was called the Sasaki metric. In 1962, Dombrowski [3] also showed at each $Z \in TM$, $TM_Z = HTM_Z \oplus VTM_Z$, where HTM_Z and VTM_Z orthogonal subspaces of dimension n, called horizontal and vertical distributions, respectively. He defined an almost Kählerian structure on TM and proved that it is Kählerian manifold if M is flat. In the same year, Tachibana and Okumura [8] showed that the tangent bundle space TM of any non-flat Riemannian space M always admits an almost Kählerian structure, which is not Kählerian. Tashiro [10] introduced a contact metric structure on the unit tangent sphere bundle T_1M and prove that contact metric structure on T_1M is K-contact iff M has constant curvature 1, in which case the structure is Sasakian.

Kowalski [5] computed the curvature tensor of Sasaki metric. Thus, on T_1M, $R(X,Y)\xi$ can be computed by the formulas for the curvature of TM.

In [1], Blair et al. introduced (κ,μ)-contact Riemannian manifolds and proved that, the tangent sphere bundle T_1M is a (κ,μ)-contact Riemannian manifold iff the base manifold M is of constant sectional curvature c.

Takahashi [9] introduced contact pseudo-metric structures (η, g), where is a contact one-form and g a pseudo-Riemannian metric associated to it, are a natural generalization of contact metric structures. Recently, contact pseudo-metric manifolds have been studied by Calvaruso and Perrone [2, 6] and authors of this paper [4] introduce and study (κ,μ)-contact pseudo-metric manifolds.

In this paper, we suppose that (M,g) is pseudo-metric manifold and define pseudo-metric on TM. Also, we introduce contact pseudo-metric structures (φ,ξ,η,G) on $T_\varepsilon M$ and prove that

$$\bar{R}(X,Y)\xi = c(2\varepsilon - c)\{\eta(Y)X - \eta(X)Y\} - 2c\{\eta(Y)\,\mathbf{h}X - \eta(X)\,\mathbf{h}Y\}$$

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if and only if the base manifold M is of constant sectional curvature. That is, the tangent sphere bundle $T_\varepsilon M$ is a (κ,μ)-contact pseudo-metric manifold iff the base manifold M is of constant sectional curvature c. Also, the contact pseudo-metric structure (φ,ξ,η,G) on $T_\varepsilon M$ is K-contact if and only if the base manifold (M,g) has constant curvature ε.

2. Preliminaries

Let (M,g) be a pseudo-metric manifold and ∇ the associated Levi-Civita connection and $R=[\nabla,\nabla]-\nabla_{[,]}$ the curvature tensor. The tangent bundle of M, denoted by TM, consists of pairs (x,u), where $x\in M$ and $u\in T_xM$,(i.e. $TM=\cup_{x\in M}T_xM$). The mapping $\pi:TM\to M, \pi(x,u)=x$ is the natural projection and for all $(x,u)\in TM$, the connection map $\mathcal{K}:TTM\to TM$ is given by $\mathcal{K}(X_*u)=\nabla_uX$, where $X:M\to TM$ is a vector field on M [3].

The tangent space $T_{(x,u)}TM$ splits into the vertical subspace $VTM_{(x,u)}$ and the horizontal subspace $HTM_{(x,u)}$ are given by $VTM_{(x,u)}:=\ker\pi_*|_{(x,u)}$ and $HTM_{(x,u)}:=\ker\mathcal{K}|_{(x,u)}$:

$$T_{(x,u)}TM=VTM_{(x,u)}\oplus HTM_{(x,u)}.$$

For every $X\in T_xM$, there is a unique vector $X^h\in HTM_{(x,u)}$, such that $\pi_*(X^h)=X$. It is called the horizontal lift of X to (x,u). Also, there is a unique vector $X^v\in VTM_{(x,u)}$, such that $X^v(df)=Xf$ for all $f\in C^\infty(M)$. X^v is called the vertical lift of X to (x,u). The maps $X\mapsto X^h$ between T_xM and $HTM_{(x,u)}$, and $X\mapsto X^v$ between T_xM and $VTM_{(x,u)}$ are isomorphisms. Hence, every tangent vector $\bar{Z}\in T_{(x,u)}TM$ can be decomposed $\bar{Z}=X^h+Y^v$ for uniquely determined vectors $X,Y\in T_xM$. The horizontal (respectively, vertical) lift of a vector field X on M to TM is the vector field X^h (respectively, X^v) on M, whose value at the point (x,u) is the horizontal (respectively, vertical) lift of X_x to (x,u).
A system of local coordinate $(x^1,\dots,x^n)$ on an open subset U of M induces on $\pi^{-1}(U)$ of TM a system of local coordinate $(\bar{x}^1,\dots,\bar{x}^n;u^1,\dots,u^n)$ as follows:

$$\bar{x}^i(x,u)=(x^i\circ\pi)(x,u)=x^i(x),\quad u^i(x,u)=dx^i(u)=ux^i$$

for $i=1,\dots,n$ and $(x,u)\in\pi^{-1}(U)$. With respect to the induced local coordinate system, the horizontal and vertical lifts of a vector field $X=X^i\frac{\partial}{\partial x^i}$ on U are given by

$$X^h=(X^i\circ\pi)\frac{\partial}{\partial\bar{x}^i}-u^b((X^a\Gamma^i_{ab})\circ\pi)\frac{\partial}{\partial u^i},\quad X^v=(X^i\circ\pi)\frac{\partial}{\partial u^i},\tag{2.1}$$

where Γ^i_{jk} are the local components of ∇, i.e., $\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^k}=\Gamma^i_{jk}\frac{\partial}{\partial x^i}$. From (2.1), one can easily calculate the brackets of vertical and horizontal lifts:

$$[X^h,Y^h]=[X,Y]^h-v\{R(X,Y)u\},\tag{2.2}$$

$$[X^h,Y^v]=(\nabla_XY)^v,\quad [X^v,Y^v]=0,\tag{2.3}$$

for all $X,Y\in\Gamma(TM)$. We use some notation, due to M. Sekizawa. Let T be a tensor field of type $(1,s)$ on M and $X_1,\dots,X_{s-1}\in\Gamma(TM)$, the vertical vector field $v\{T(X_1,\dots,u,\dots,X_{s-1})\}$ on $\pi^{-1}(U)$ is given by

$$v\{T(X_1,\dots,u,\dots,X_{s-1})\}:=u^a(T(X_1,\dots,\frac{\partial}{\partial x^a},\dots,X_{s-1}))^v.$$

If f is a smooth function on M and X is a vector field on M, then

$$X^h(f\circ\pi)=(Xf)\circ\pi,\quad X^v(f\circ\pi)=0. \tag{2.4}$$

In particular, we write $X=X^i\frac{\partial}{\partial x^i}$ on U, and then we have

$$X^h(\bar{x}^i)=X^i\circ\pi,\quad X^v(\bar{x}^i)=0. \tag{2.5}$$

Further, from (2.1), we have

$$X^h(u^i)=-u^b(X^a\Gamma^i_{ab})\circ\pi,\quad X^v(u^i)=X^i\circ\pi. \tag{2.6}$$

Let (M,g) be a pseudo-metric manifold. On the tangent bundle TM, we can define a pseudo-metric $\tilde{g}$ to be

$$\tilde{g}(X^h,Y^h)=\tilde{g}(X^v,Y^v)=g(X,Y)\circ\pi,\quad \tilde{g}(X^h,Y^v)=0 \tag{2.7}$$

for all $X,Y\in\Gamma(TM)$. We call it Sasaki pseudo-metric. According (2.7), If $\{E_1,\dots,E_n\}$ is an orthonormal frame field on U then $\{E_1^v,\dots,E_n^v,E_1^h,\dots,E_n^h\}$ is an orthonormal frame field on $\pi^{-1}(U)$. So, we have the following:

Proposition 2.1. *If the index of g is ν then the index of the Sasaki pseudo-metric $\tilde{g}$ is 2ν.*

3. The curvature of the unit tangent sphere bundle with pseudo-metric

Let $(TM,\tilde{g})$ be the tangent bundle of (M,g) endowed with its Sasaki pseudo-metric. We consider the hypersurface $T_\varepsilon M=\{(x,u)\in TM|g_x(u,u)=\varepsilon\}$, which we call the unit tangent sphere bundle. A unit normal vector field N on $T_\varepsilon M$ is the (vertical) vector field $N=u^i\frac{\partial}{\partial u^i}=u^i(\frac{\partial}{\partial x^i})^v$. N is independent of the choice of local coordinates and it is defined globally on TM. We introduce some more notation. If $X\in T_xM$, we define the tangential lift of X to $(x,u)\in T_\varepsilon M$ by

$$X^t_{(x,u)}=X^v_{(x,u)}-\varepsilon g(X,u)N_{(x,u)}. \tag{3.1}$$

Clearly, the tangent space to $T_\varepsilon M$ at (x,u) is spanned by vectors of the form X^h and X^t, where $X\in T_xM$. Note that $u^t_{(x,u)}=0$. The tangential lift of a vector field X on M to $T_\varepsilon M$ is the vertical vector field X^t on $T_\varepsilon M$, whose value at the point $(x,u)\in T_\varepsilon M$ is the tangential lift of X_x to (x,u). For a tensor field T of type $(1,s)$ on M and $X_1,\dots,X_{s-1}\in\Gamma(TM)$, we define the vertical vector fields $t\{T(X_1,\dots,u,\dots,X_{s-1})\}$ and $t\{T(X_1,\dots,u,\dots,u,\dots,X_{s-2})\}$ on $T_\varepsilon M$ in the obvious way.

In what follows, we will give explicit expressions for the brackets of vector fields on $T_\varepsilon M$ involving tangential lifts, the Levi-Civita connection and the associated curvature tensor of the induced metric $\bar{g}$ on $T_\varepsilon M$.

First, for the brackets of vector fields on $T_\varepsilon M$ involving tangential lifts, we obtain

$$[X^h,Y^t]=(\nabla_XY)^t,\quad [X^t,Y^t]=\varepsilon g(X,u)Y^t-\varepsilon g(Y,u)X^t. \tag{3.2}$$

Next, we denote by $\bar{g}$ the pseudo-metric induced on $T_\varepsilon M$ from $\tilde{g}$ on TM as follows:

$$\begin{aligned}\bar{g}(X^h,Y^h)&=g(X,Y),\quad \bar{g}(X^h,Y^t)=0\\ \bar{g}(X^t,Y^t)&=g(X,Y)-\varepsilon g(X,u)g(Y,u),\end{aligned} \tag{3.3}$$

4. The contact pseudo-metric structure of the unit tangent sphere bundle

First, we give some basic facts on contact pseudo-metric structures. A pseudo-Riemannian manifold (M^{2n+1}, g) has a contact pseudo-metric structure if it admits a vector field ξ, a one-form η and a $(1,1)$-tensor field φ satisfying

$$\begin{aligned} &\eta(\xi) = 1, \quad \varphi^2(X) = -X + \eta(X)\xi, \\ &g(\varphi X, \varphi Y) = g(X, Y) - \varepsilon\eta(X)\eta(Y), \quad d\eta(X, Y) = g(X, \varphi Y), \end{aligned} \tag{4.1}$$

where $\varepsilon = \pm 1$ and $X, Y \in \Gamma(TM)$. In this case, $(M, \varphi, \xi, \eta, g)$ is called a contact pseudo-metric manifold. In particular, the above conditions imply that the characteristic curves, i.e., the integral curves of the characteristic vector field ξ, are geodesics.
If ξ is in addition a Killing vector field with respect to g, then the manifold is said to be a K-contact (pseudo-metric) manifold. Another characterizing property of such contact pseudo-metric manifolds is the following:
geodesics which are orthogonal to ξ at one point, always remain orthogonal to ξ. This yields a second special class of geodesics, the φ-geodesics.
Next, if $(M^{2n+1}, \varphi, \xi, \eta, g)$ is a contact pseudo-metric manifold satisfying the additional condition $N_\varphi(X, Y) + 2d\eta(X, Y)\xi = 0$ is said to be Sasakian, where $N_\varphi(X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]$ is the Nijenhuis torsion tensor of φ. A contact pseudo-metric structure is a Sasakian structure iff R satisfies

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{4.2}$$

In particular, one can show that the characteristic vector field ξ is a Killing vector field. Hence, a Sasakian manifold is also a K-contact manifold. In a contact pseudo-metric manifold $M^{2n+1}(\varphi, \xi, \eta, g)$, defined the $(1,1)$-tensor field $\mathbf{h}$ by $\mathbf{h}X = \frac{1}{2}(L_\xi\varphi)(X)$, where L denotes the Lie derivative. The tensors $\mathbf{h}$ is self-adjoint operator satisfying([2, 6])

$$\mathbf{h}\varphi = -\varphi\,\mathbf{h}, \quad \mathbf{h}\xi = 0, \quad \nabla_X\xi = -\varepsilon\varphi X - \varphi\,\mathbf{h}X. \tag{4.3}$$

(see [2, 6] for more details). If a contact pseudo-metric manifold satisfying

$$R(X, Y)\xi = \varepsilon\kappa\big(\eta(Y)X - \eta(X)Y\big) + \varepsilon\mu\big(\eta(Y)\,\mathbf{h}X - \eta(X)\,\mathbf{h}Y\big),$$

we call (κ, μ)-contact pseudo-metric manifold, where $(\kappa, \mu) \in \mathbb{R}^2$. the (κ, μ)-contact pseudo-metric manifold is Sasakian iff $\kappa = \varepsilon$ and hence $\mathbf{h} = 0$, by (4.2). (see [4] for more details).
Take now an arbitrary pseudo-metric manifold (M, g). One can easily define an almost complex structure J on TM in the following way:

$$JX^h = X^v, \quad JX^v = -X^h \tag{4.4}$$

for all vector fields X on M. From (2.2), and (2.3), we have the almost complex structure J is integrable if and only if (M, g) is flat. From the definition (2.7) of the pseudo-metric $\tilde{g}$ on TM, it follows immediately that $(TM, \tilde{g}, J)$ is almost Hermitian. Moreover, J defines an almost Kählerian structure. It is a Kähler manifold only when (M, g) is flat[3].
Next, we consider the unit tangent sphere bundle $(T_\varepsilon M, \bar{g})$, which is isometrically embedded as a hypersurface in $(TM, \tilde{g})$ with unit normal field N. Using the almost

complex structure J on TM, we define a unit vector field ξ', a one-form η' and a $(1,1)$-tensor field φ' on $T_\varepsilon M$ by

$$\xi' = -JN, \quad JX = \varphi' X + \eta'(X)N. \tag{4.5}$$

In local coordinates, ξ', η' and φ' are described by

$$\begin{aligned} &\xi' = u^i(\frac{\partial}{\partial x^i})^h, \quad \eta'(X^t) = 0, \quad \eta'(X^h) = \varepsilon g(X,u), \\ &\varphi'(X^t) = -X^h + \varepsilon g(X,u)\xi', \quad \varphi'(X^h) = X^t, \end{aligned} \tag{4.6}$$

where $X, Y \in \Gamma(TM)$. It is easily checked that these tensors satisfy the conditions (4.1) excepts or the last one, where we find $\varepsilon\bar{g}(X, \varphi' Y) = 2d\eta'(X,Y)$. So strictly speaking, $(\varphi', \xi', \eta', \bar{g})$ is not a contact pseudo-metric structure. Of course, the difficulty is easily rectified and

$$\eta = \frac{1}{2}\eta', \quad \xi = 2\xi', \quad \varphi = \varepsilon\varphi', \quad G = \frac{1}{4}\bar{g}$$

is taken as the standard contact pseudo-metric structure on $T_\varepsilon M$.

Theorem 4.1. *The tangent sphere bundle $T_\varepsilon M$ is (κ,μ)-contact pseudo-metric manifold if and only if the base manifold M is of constant sectional curvature c and $\kappa = \varepsilon c(2\varepsilon - c), \mu = -2\varepsilon c$.*

Theorem 4.2. *The contact pseudo-metric structure (φ, ξ, η, G) on $T_\varepsilon M$ is K-contact if and only if the base manifold (M,g) has constant curvature ε, in which case the structure on $T_\varepsilon M$ is Sasakian.*

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